Einstein static universe as a brane in extra dimensions

A. Gruppuso[a,b,c*], E. Roessl[a†] and M. Shaposhnikov[a‡]

[a] *Institut de théorie des phénomènes physiques (ITP)*
Laboratoire de Physique des Particules et Cosmologie (LPPC)
École polytechnique fédérale de Lausanne
CH-1015 Lausanne, Switzerland

[b] *IASF/CNR, Istituto di Astrofisica Spaziale e Fisica Cosmica*
Sezione di Bologna
Consiglio Nazionale delle Ricerche
via Gobetti 101, I-40129 Bologna - Italy
and
[c] *Dipartimento di Fisica, Università di Bologna and I.N.F.N., Sezione di Bologna,*
via Irnerio 46, 40126, Bologna, Italy

February 1, 2008

Abstract

We present a brane-world scenario in which two regions of AdS_5 space-time are glued together along a 3-brane with constant positive curvature such that *all* spatial dimensions form a compact manifold of topology S^4. It turns out that the induced geometry on the brane is given by Einstein's static universe. It is possible to achieve an anisotropy of the manifold which allows for a huge hierarchy between the size of the extra dimension R and the size of the observable universe R_U at present. This anisotropy is also at the origin of a very peculiar property of our model: the physical distance between *any two points* on the brane is of the order of the size of the extra dimension R regardless of their distance measured with the use of the induced metric on the brane. In an intermediate distance regime $R \ll r \ll R_U$ gravity on the brane is shown to be effectively 4-dimensional, with corresponding large distance corrections, in complete analogy with the Randall-Sundrum II model. For very large distances $r \sim R_U$ we recover gravity in Einstein's static universe. However, in contrast to the Randall-Sundrum II model the difference in topology has the advantage of giving rise to a geodesically complete space.

*gruppuso@bo.iasf.cnr.it
†ewald.roessl@epfl.ch
‡mikhail.shaposhnikov@epfl.ch

1 Introduction

Recent suggestions that large [1]–[3] or infinite [4]–[6] extra dimensions are not necessarily in conflict with present observations provide new opportunities for addressing several outstanding problems of modern theoretical physics like the hierarchy problem [1]–[3],[5], [7] or the cosmological constant problem [8]–[10]. In the course of this development and inspired by string theoretical arguments [11], the notion of brane world scenarios emerged in which the usual Standard Model fields are supposed to be confined to a so-called 3-brane, a 4-dimensional sub-manifold of some higher-dimensional space-time. As shown in [6] also gravity can appear to be effectively 4-dimensional for a brane-bound observer provided the conventional scheme of Kaluza-Klein compactification [12]–[15] is replaced by a compactification using non-factorizable (also called warped) geometries (see also [8]). This triggered an immense research activity in theories involving 3-branes with interests ranging from elucidating the global space-time structure of brane world scenarios, properties of gravity, cosmology and brane cosmological perturbations, generalizations to higher dimensions etc. While the possibilities are rich, realistic scenarios remain rare. For example the simple Randall-Sundrum-II model [6] faces the problem of being geodesically incomplete [16, 17, 18]. It is therefore reasonable to look for adequate alternatives or generalizations to the Randall-Sundrum-II model which avoid the above mentioned problems while sharing its pleasant feature of the effective 4-dimensional low energy gravity on the brane.

In this paper we present a 5-dimensional brane-world model which solves the geodesic incompleteness of the Randall-Sundrum II model while preserving its phenomenological properties concerning the localization of gravity.

We try to illustrate our motivation for considering a particular geometry by using the simple picture of a domain structure in extra dimensions resulting from a spontaneously broken discrete symmetry. The associated Higgs-field takes different values in regions separated by a domain wall, which restricts the possibilities of combining domain walls depending on the global topology of the space-time under consideration. We concentrate on the 5-dimensional case, see [7],[19]–[24] for higher dimensional constructions.

Let us analyze a few simple cases:

- Non-compact extra dimension: $y \in (-\infty, +\infty)$. In this case, by choosing the origin $y = 0$ to coincide with the position of the brane, we obtain two non-overlapping regions $(-\infty, 0)$ and $(0, +\infty)$ and we can thus consistently have one brane in such a theory. An example for this configuration is provided by the Randall-Sundrum II model [6].
- Compact extra dimension: $y \in [0, 2\pi]$. We now consider two possibilities, depending on the spatial topology of our manifold:
  - **(a)** $\mathbb{R}^3 \times S^1$: If the ordinary dimensions are supposed to be non-compact, it is not possible to consistently put only one brane in the extra dimensions. At least two branes are needed.
  - **(b)** S^4: If *all* spatial coordinates are part of a compact manifold, the simple picture shown in Fig. 1 seems to suggest that it is possible to consistently put a single brane in the bulk space-time.

To the best of our knowledge case **(b)** has not yet been considered in the literature and our aim is to present such a construction.

This paper is organized as follows: in Section 2 we discuss the basic geometric and topological properties of our brane-world scenario like Einstein equations, junction conditions and the distance hierarchy between the extra dimension R and the observable universe R_U. Section 3 is dedicated to the study of geodesics and the demonstration that our model does not suffer from being geodesically incomplete. In section 4 we present a detailed computation of the propagator of a massless scalar field in the given background serving as an easy, phenomenological approach to the study of gravity. We discuss the behavior of the two-point function in three different distance regimes. It turns out that the computations are rather technical and we therefore collect large parts of it in four appendices. We eventually draw conclusions in section 5.

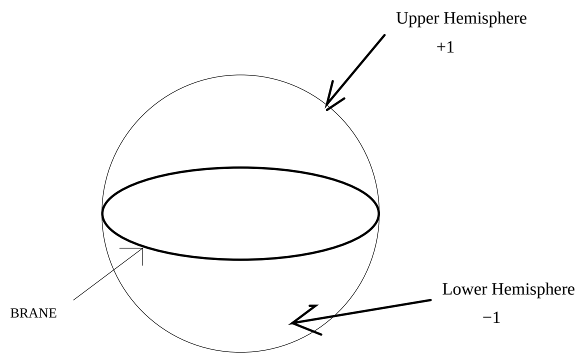

Figure 1: Compact topology. S^3-brane embedded in S^4. $+1$ and -1 symbolically represent the different vacuum expectation values taken by the Higgs field in different regions of space.

2 The background equations

The aim of this section is to present the topology and geometry of the brane-world model which we set out to study but also to motivate how this particular model emerged through imposing several physical conditions on the more general set of solutions.

2.1 Einstein equations

We would like to construct a space-time with all spatial dimensions being part of the same compact manifold. One more motivation for this lies in a possible solution of the strong CP problem within theories with extra dimensions [25]. The overall topology would then be given by $\mathbb{R} \times \mathbb{K}$, where $\mathbb{R}$ represents the time-coordinate and $\mathbb{K}$ any compact manifold. As announced, we will restrict ourselves to the case of one extra dimension and an induced metric on the brane characterized by a spatial component of geometry S^3. The idea is to combine two 5-dimensional regions of space-time dominated by a cosmological constant Λ in such a way that the border of the two regions can be identified with a 3-brane, constituting our observable universe. As pointed out in [25] the manifold $\mathbb{K}$ has to be highly anisotropic in order to single out the small extra dimension from the three usual ones. It is not a priory clear whether the Einstein equations allow for such solutions at all and if so whether gravity can be localized on the brane in such a setup.

We choose the following ansatz for the 5-dimensional metric consistent with the above requirements:[1]

$$ds^2 = g_{MN} dx^M dx^N = -\sigma^2(\theta)\, dt^2 + R_U^2\, \gamma^2(\theta)\, d\Omega_3^2 + R^2\, d\theta^2\,, \tag{2.1}$$

where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ denotes the extra coordinate and R and R_U are constants representing the size of the extra dimension and the size of the observable universe at present, respectively. $d\Omega_3^2$ denotes the line element of a 3-sphere:

$$d\Omega_3^2 = d\varphi_1^2 + \sin^2\varphi_1\, d\varphi_2^2 + \sin^2\varphi_1\, \sin^2\varphi_2\, d\varphi_3^2\,, \tag{2.2}$$

where φ_1, φ_2 belong to the interval $[0, \pi]$, φ_3 to $[0, 2\pi]$. Capital latin letters $M, N, ..$ will range from 0 to 4. In order to obtain a compact space, we require $\gamma(\pm\frac{\pi}{2}) = 0$.

Metrics which can locally be put into the form (2.1) are sometimes referred to as *asymmetrically warped* metrics due to two different functions (σ and γ in our notations) multiplying the temporal and spatial part of the 4-dimensional coordinate differentials. Their relevance in connection with 4-dimensional Lorentz-violation at high energies was first pointed out in [27]. More recent discussions of this subject can be found in [28]–[29] or [16] and references therein. A common prediction of theories of this kind is that dispersion relations get modified at high energies. For a field-theoretical discussion of Lorentz violating effects in the context of the Standard Model of particle physics see [30].

The Einstein equations in 5 dimensions with a bulk cosmological constant Λ and a stress-energy tensor T_{MN} take the following form:

$$R_{MN} - \frac{1}{2}\, R\, g_{MN} + \Lambda\, g_{MN} = \frac{8\pi}{M^3}\, T_{MN}\,, \tag{2.3}$$

where M is the fundamental scale of gravity. We choose to parametrize the stress-energy tensor in the following way, consistent with the symmetries of the metric:

$$T^0{}_0 = \epsilon_0(\theta) < 0\,, \quad T^i{}_j = \delta^i_j\, \epsilon(\theta)\,, \quad T^\theta{}_\theta = \epsilon_\theta(\theta)\,, \tag{2.4}$$

where as indicated the above diagonal components depend only on the extra coordinate θ. Lower case latin indices i, j label the coordinates on S^3.

Using the metric ansatz (2.1) together with the stress-energy tensor (2.4) the Einstein equations (2.3) become

$$\frac{3}{R^2}\left[\left(\frac{\gamma'}{\gamma}\right)^2 + \frac{\gamma''}{\gamma} - \left(\frac{R}{R_U}\right)^2 \frac{1}{\gamma^2}\right] + \Lambda = \frac{8\pi}{M^3}\epsilon_0\,, \tag{2.5}$$

$$\frac{1}{R^2}\left[\left(\frac{\gamma'}{\gamma}\right)^2 + 2\frac{\gamma'}{\gamma}\frac{\sigma'}{\sigma} + 2\frac{\gamma''}{\gamma} + \frac{\sigma''}{\sigma} - \left(\frac{R}{R_U}\right)^2 \frac{1}{\gamma^2}\right] + \Lambda = \frac{8\pi}{M^3}\epsilon\,, \tag{2.6}$$

$$\frac{3}{R^2}\left[\left(\frac{\gamma'}{\gamma}\right)^2 + \frac{\gamma'}{\gamma}\frac{\sigma'}{\sigma} - \left(\frac{R}{R_U}\right)^2 \frac{1}{\gamma^2}\right] + \Lambda = \frac{8\pi}{M^3}\epsilon_\theta\,, \tag{2.7}$$

where $'$ denotes differentiation with respect to θ. The conservation of stress-energy, or equivalently, the Bianchi-identities lead to the following constraint relating the three independent components ϵ_0, ϵ and ϵ_θ:

$$\epsilon_\theta' + \left(\frac{\sigma'}{\sigma} + 3\frac{\gamma'}{\gamma}\right)\epsilon_\theta - \frac{\sigma'}{\sigma}\,\epsilon_0 - 3\,\frac{\gamma'}{\gamma}\epsilon = 0\,. \tag{2.8}$$

[1]Our conventions for the metric and the Riemann tensor are those of reference [26].

2.2 Vacuum solution

In this paper we do not intend to provide a field theoretical model which could generate the geometry we are about to describe. Our aim is to study a singular brane, located at $\theta_b = 0$ separating two vacuum regions in the bulk. Of course the solutions to the Einstein equations in vacuum with a cosmological constant term are nothing but the familiar de-Sitter ($\Lambda > 0$), Minkowski ($\Lambda = 0$) and anti-de-Sitter ($\Lambda < 0$) space-times. To see how these geometries can be recovered using the line element (2.1) we set the right hand sides of (2.5)-(2.7) equal to zero and subtract eq. (2.7) from eq. (2.5) to obtain $\sigma = c\,\gamma'$, where c is a constant. Putting this back in eqs. (2.5)-(2.7), we are left with only one independent differential equation

$$\frac{3}{R^2}\left[\left(\frac{\gamma'}{\gamma}\right)^2 + \frac{\gamma''}{\gamma} - \left(\frac{R}{R_U}\right)^2 \frac{1}{\gamma^2}\right] + \Lambda = 0. \tag{2.9}$$

The solution of (2.9) is trivial and by means of $\sigma = c\,\gamma'$ we find

$$\gamma(\theta) = \frac{\sinh\left[\omega\left(\frac{\pi}{2} - |\theta|\right)\right]}{\sinh(\omega\frac{\pi}{2})}\,, \qquad \sigma(\theta) = \frac{\cosh\left[\omega\left(\frac{\pi}{2} - |\theta|\right)\right]}{\cosh(\omega\frac{\pi}{2})}\,, \tag{2.10}$$

provided that $R_U = R\sinh(\omega\frac{\pi}{2})/\omega$ and $\omega^2 = -\Lambda R^2/6$. Notice that the solution (2.10) does not contain any integration constant because we already imposed the boundary conditions $\gamma(\pm\frac{\pi}{2}) = 0$ and $\gamma(0) = \sigma(0) = 1$. Moreover, we chose the whole setup to be symmetric under the transformation $\theta \to -\theta$. From (2.10) it is now obvious that locally (for $\theta > 0$ and $\theta < 0$) the line element (2.1) correctly describes de-Sitter, Minkowski and anti-de-Sitter space-times for imaginary, zero and real values of ω, respectively. For our purposes only the AdS solution will be of any interest as we will see in the following section. A simple change of coordinates starting from (2.1) and (2.10) shows that the AdS-radius in our notations is given by $R_{AdS} = R/\omega = \sqrt{-6/\Lambda}$.[2] At this stage, the validity of the vacuum solution (2.10) is restricted to the bulk, since it is not even differentiable in the classical sense at $\theta = 0$. In order to give sense to (2.10) for all values of θ we will have to allow for some singular distribution of stress-energy at $\theta = 0$ and solve (2.5)-(2.7) in the sense of distributions.

2.3 The complete solution for a singular brane

As announced, we now refine our ansatz for the energy momentum tensor (2.4) to allow for a solution of eqs. (2.5)-(2.7) in the whole interval $-\pi/2 \le \theta \le \pi/2$:

$$\epsilon_0(\theta) = c_0\,\frac{\delta(\theta)}{R}\,, \quad \epsilon(\theta) = c\,\frac{\delta(\theta)}{R}\,, \quad \epsilon_\theta(\theta) = c_\theta\,\frac{\delta(\theta)}{R}\,. \tag{2.11}$$

After replacing the above components (2.11) in eqs. (2.5)-(2.7) and integrating over θ from $-\eta$ to η, followed by the limit $\eta \to 0$ we find:

$$:\frac{\gamma'}{\gamma}: \;=\frac{8\pi R}{3M^3}c_0\,, \tag{2.12}$$

$$:\frac{\sigma'}{\sigma}: + 2:\frac{\gamma'}{\gamma}: \;=\frac{8\pi R}{M^3}c\,, \tag{2.13}$$

$$0 = c_\theta\,, \tag{2.14}$$

[2] See also footnote 7.

where the symbol $:\ldots:$ is used to denote the jump of a quantity across the brane defined by:

$$: f : \equiv \lim_{\eta \to 0} \left[f(\eta) - f(-\eta) \right]. \tag{2.15}$$

Note that in the above step we made use of the identity

$$\frac{\gamma''}{\gamma} = \left(\frac{\gamma'}{\gamma}\right)' + \left(\frac{\gamma'}{\gamma}\right)^2 \tag{2.16}$$

together with the continuity of γ on the brane. Specifying (2.12) and (2.13) to (2.10) we have:

$$c_0 = -\frac{3}{4\pi} M^3 \frac{\omega}{R} \coth\left(\omega \frac{\pi}{2}\right), \tag{2.17}$$

$$c \ = -\frac{1}{4\pi} M^3 \frac{\omega}{R} \left[\tanh\left(\omega \frac{\pi}{2}\right) + 2\coth\left(\omega \frac{\pi}{2}\right)\right], \tag{2.18}$$

$$c_\theta = 0\,. \tag{2.19}$$

Eqs. (2.17) and (2.18) relate the energy-density and the pressure of the singular brane to the bulk cosmological constant Λ (via ω), the size of the extra dimension R and the fundamental scale of gravity M. With the above relations (2.17)-(2.19) we can now interpret (2.10) as a solution to the Einstein equations (2.5)-(2.7) in the sense of distributions. Also the stress-energy conservation constraint (2.8) is satisfied based on the identity $\delta(x)\,\mathrm{sign}(x) = 0$ again in the distributional sense.

Note that $c_0/c \to 1$ in the limit $\omega \to \infty$. Moreover, for larger and larger values of ω, c_0 and c approach the brane tension of the Randall-Sundrum II model and the above eqs. (2.17) and (2.18) merge to the equivalent relation in the Randall-Sundrum II case. This is no surprise since taking the limit $\omega \to \infty$ corresponds to inflating and flattening the 3-brane so that we expect to recover the case of the flat Randall-Sundrum II brane.

We finish this section by the discussion of some physical properties of our manifold. We first observe that its spatial part is homeomorphic to a 4-sphere S^4. This is obvious from the metric (2.1) and the explicit expression for γ given in (2.10). Geometrically, however, our manifold differs from S^4 due to the high anisotropy related to the smallness of the extra dimension. The ratio of typical distance scales in the bulk and on the brane is given by

$$\frac{R}{R_U} = \frac{\omega}{\sinh\left(\frac{\omega\pi}{2}\right)}\,. \tag{2.20}$$

It is now clear that the above ratio (2.20) can only be made very small in the case of real ω (AdS-space-time). For the size of the observable universe we take the lower bound $R_U > 4\,\mathrm{Gpc} \sim 10^{28}\,\mathrm{cm}$ while the size of the extra dimension is limited from above [31]: $R < 10^{-2}\,\mathrm{cm}$, leaving us with $\omega > 50$.

Finally we would like to point out a very peculiar property of the manifold under consideration: as it can immediately be deduced from the line element (2.1) and (2.10), any two points on the brane are separated by not more than a distance of the order of R regardless of their distance as measured by an observer on the brane using the induced metric.

3 Geodesics

It is well known that the Randall Sundrum-II model is timelike and lightlike geodesically incomplete [16, 17, 18] which means that there exists inextendible timelike and lightlike geodesics.[3] An

[3]For a precise definition of a geodesically incomplete space see e.g. [32].

inextendible geodesic is a geodesic parametrized by an affine parameter τ such that by using up only a finite amount of affine parameter the geodesic extends over infinite coordinate distances. In a more physical language one could reformulate the above statement by saying that it takes only a finite amount of affine parameter τ in order to reach the infinities of the incomplete space-time. As we will illustrate later in this chapter, the reason why the Randall-Sundrum II setup ceases to be geodesically complete is simply due to a specific way of gluing two patches of AdS_5. One of the main motivations for this work was to provide an alternative to the Randall Sundrum II model that has the advantage of being geodesically complete while conserving the pleasant phenomenological features of the latter. We divide the discussion of geodesics in two parts: in subsection 3.1 we illustrate the effects of incomplete geodesics in the Randall-Sundrum II setup for timelike and lightlike geodesics. In the following subsection 3.2 we demonstrate why our setup is geodesically complete by looking at corresponding geodesics. Finally, we complement the discussions by illustrating the physics with the use of the Penrose-diagram of (the universal covering space-time of) AdS_5.

3.1 Geodesics in the Randall-Sundrum II setup

Our discussion of geodesics in this chapter is in no sense meant to be complete. Without going into the details of the computations we merely intend to present the solutions of the geodesic equations in certain cases. For more general and more complete discussions of this issue we refer to the literature, see e.g. [17, 33] and references therein. We first consider lightlike geodesics in the Randall-Sundrum II background metric given in appendix A, eq. (A.1). Let us suppose that a photon is emitted at the brane at $y = 0$ in the positive y-direction at coordinate time $t = 0$ then reflected at $y = y_1$ at the time $t = t_1$ to be observed by an observer on the brane at time $t = t_2$. In this situation t corresponds to the proper time of an observer on the brane at rest. A simple calculation reveals

$$t_2 = 2t_1 = \frac{2}{k}\left(e^{ky_1} - 1\right). \tag{3.1}$$

A brane bound observer will therefore note that it takes an infinite time for a photon to escape to $y = \infty$. However, parameterizing the same geodesic by an affine parameter τ reveals the lightlike incompleteness of the Randall-Sundrum II space-time. Let the events of emission, refection (at $y = y_1$) and arrival on the brane again be labeled by $\tau = 0$, $\tau = \tau_1$ and $\tau = \tau_2$, respectively. Using the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0\,, \tag{3.2}$$

an easy computation shows:

$$\tau_2 = 2\tau_1 = \frac{2}{ck}\left(1 - e^{-ky_1}\right). \tag{3.3}$$

The constant c is a remnant of the freedom in the choice of an affine parameter.[4] From the last equation we see that now

$$\lim_{y_1\to\infty} \tau_1 = \frac{1}{ck}, \tag{3.4}$$

meaning that in order to reach infinity ($y = \infty$) in the extra dimension it takes only a finite amount $1/(ck)$ of affine parameter τ, the expression of incompleteness of the Randall-Sundrum II space-time with respect to affinely parametrized lightlike geodesics.

[4] In general, two affine parameters λ and τ are related by $\lambda = c\tau + d$, since this is the most general transformation leaving the geodesic equation (3.2) invariant. The choice of the origin of time to coincide with the emission of the photon only fixes d but does not restrict c.

In the case of timelike geodesics the inconsistency is even more striking. As shown in e.g. [18], a massive particle starting at the brane with vanishing initial velocity travels to $y = \infty$ in finite proper time given by $\tau_p = \pi/(2k)$ while for a brane-bound observer this happens in an infinite coordinate time. To summarize, the Randall-Sundrum II brane-world model is geodesically incomplete both for null and for timelike geodesics. As we will see in the next section, the geodesic incompleteness is a direct consequence of the use of a particular coordinate system in AdS_5, the so-called Poincaré-coordinate system, which covers only a part of the full AdS_5 space-time. We will also see that the problem of incomplete geodesics is absent in the setup we propose in this paper.

3.2 Geodesics in the background (2.1) and Penrose-diagram

We would now like to answer similar questions to the ones considered in the previous section for the background (2.1). For example, we would like to know what time it takes for light to travel from the brane (at θ=0) in the θ-direction to a given point in the upper hemisphere with θ coordinate θ_1, to be reflected and to return to the brane. As in the last section, t and τ will denote the coordinate time (proper time of a stationary observer on the brane) and the affine parameter used for parameterizing the geodesics, respectively. Again we choose $t = 0$ ($\tau = 0$) for the moment of emission, t_1 (τ_1) for the reflection and t_2 (τ_2) for the time where the photon returns to the brane. Due to the enormous hierarchy of distance scales in our model, one might wonder whether photons (or gravitons) are able to carry information from an arbitrary point on the brane to any other point on the brane connected to the first one by a null-geodesic in the extra dimension. For an observer on the brane such a possibility would be interpreted as 4-dimensional causality-violation. However, as we will see in the following, none of these possibilities exist in our model. Omitting all details we find

$$t_2 = 2t_1 = 4R_U \coth\left(\frac{\omega\pi}{2}\right) \arctan\left[\frac{\sinh(\frac{\omega\theta_1}{2})}{\cosh\left[\frac{\omega}{2}\left(\pi - \theta_1\right)\right]}\right], \tag{3.5}$$

so that an observer on the brane will see that the photon reaches the "north pole" $\theta_1 = \pi/2$ at finite time

$$t_1 = R_U \coth\left(\frac{\omega\pi}{2}\right) \arctan\left[\sinh\left(\frac{\omega\pi}{2}\right)\right] \approx R_U \frac{\pi}{2}. \tag{3.6}$$

Note that due to the warped geometry, it is R_U entering the last relation and not R, so that even though the physical distance to the "north pole" is of the order of R it takes a time of the order of R_U for photons to reach it, excluding causality violation on the brane as discussed above. If the same geodesic is parametrized using an affine parameter we obtain

$$\tau_2 = 2\tau_1 = \frac{2}{c}\frac{R}{\omega}\left[\tanh\left(\frac{\omega\pi}{2}\right) - \frac{\sinh\left[\omega\left(\frac{\pi}{2} - \theta_1\right)\right]}{\cosh\frac{\omega\pi}{2}}\right], \tag{3.7}$$

such that the amount of affine parameter needed to reach $\theta_1 = \pi/2$ starting from the brane is:

$$\Delta\tau = \frac{1}{c}\frac{R}{\omega}\tanh\left(\frac{\omega\pi}{2}\right) \approx \frac{1}{c}\frac{R}{\omega}. \tag{3.8}$$

Here again c reflects the freedom in the choice of the affine parameter.

The results formally resemble those of the Randall-Sundrum II case. However, the important difference is that in our case each geodesic can trivially be extended to arbitrary values of the affine parameter, a simple consequence of the compactness of our space. Once the photon reaches the point

$\theta = 0$ it continues on its geodesic, approaching the brane, entering the southern "hemisphere", etc. It is clear that it needs an infinite amount of affine parameter in order to travel infinite coordinate distances. Therefore, the null geodesics in our setup which are the analogues of the incomplete null geodesics in the Randall-Sundrum II setup turn out to be perfectly complete due to the compactness of our space. The situation for timelike geodesics is fully analog to the case of the null geodesics.

To end this section about the geometric properties of our model we would like to discuss the conformal structure of our space-time and point out differences to the Randall-Sundrum II setup. Let us review briefly the basic properties of AdS_5 space-time to the extend that we will need it in the following discussion.[5] AdS_5 space-time can be thought of as the hyperboloid defined by

$$X_0^2 + X_5^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = a^2 \tag{3.9}$$

embedded in a flat space with metric

$$ds^2 = -dX_0^2 - dX_5^2 + dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2, \tag{3.10}$$

a being the so-called AdS-radius. The *global coordinates* of AdS_5 are defined by

$$\begin{aligned} X_0 &= a\cosh\chi\cos\tau, & X_5 &= a\cosh\chi\sin\tau, \\ X_1 &= a\sinh\chi\cos\varphi_1, & X_2 &= a\sinh\chi\sin\varphi_1\cos\varphi_2, \\ X_3 &= a\sinh\chi\sin\varphi_1\sin\varphi_2\cos\varphi_3, & X_4 &= a\sinh\chi\sin\varphi_1\sin\varphi_2\sin\varphi_3, \end{aligned} \tag{3.11}$$

where the coordinates are confined by $0 \le \chi$, $-\pi \le \tau \le \pi$, $0 \le \varphi_1 \le \pi$, $0 \le \varphi_2 \le \pi$, $0 \le \varphi_3 \le 2\pi$ and $\tau = -\pi$ is identified with $\tau = \pi$. These coordinates cover the full hyperboloid exactly once. Allowing τ to take values on the real line without the above identification of points gives the universal covering space $CAdS_5$ of AdS_5.[6] In these coordinates, the line element (3.10) can be written:[7]

$$ds^2 = a^2\left(-\cosh^2\chi\, d\tau^2 + d\chi^2 + \sinh^2\chi\, d\Omega_3^2\right) . \tag{3.12}$$

Another coordinate system can be defined by

$$\begin{aligned} X_0 &= \frac{1}{2u}\left[1 + u^2\left(a^2 + \vec{x}^2 - \bar{t}^2\right)\right], & X_5 &= a\,u\,\bar{t}, \\ X^i &= a\,u\,x^i,\ i = 1,2,3\ , & X^4 &= \frac{1}{2u}\left[1 - u^2\left(a^2 - \vec{x}^2 + \bar{t}^2\right)\right], \end{aligned} \tag{3.13}$$

with $u > 0$, $\bar{t} \in (-\infty, \infty)$ and $x^i \in (-\infty, \infty)$. In these *Poincaré coordinates* the line element (3.10) takes the form

$$ds^2 = a^2\left[\frac{du^2}{u^2} + u^2\left(-d\bar{t}^2 + d\vec{x}^2\right)\right]. \tag{3.14}$$

In contrast to the global coordinates, the Poincaré coordinates do not cover the whole of the AdS_5 and $CAdS_5$ space-times [34]. From (3.14), after changing coordinates according to $dy = -a\,du/u$ and rescaling t and x^i by the AdS-radius a we recover the original Randall-Sundrum II coordinate

[5] We mainly follow [34].

[6] Whenever we used the word AdS_5 so far in this paper we actually meant $CAdS_5$. For reasons of clarity, however, we will brake with this common practice in the rest of this section.

[7] It is obvious that the metric (2.1) for $0 \le \theta \le \pi/2$ ($-\pi/2 \le \theta \le 0$) reduces to the above line-element (3.12) under the following coordinate transformation: $\chi = \omega(\frac{\pi}{2} \mp \theta)$, $\tau = t\omega/\left[R\cosh\left(\frac{\omega\pi}{2}\right)\right]$, together with $a = R/\omega$.

system given in (A.1). The restrictions on y in the Randall-Sundrum II setup further limit the range covered by their coordinate system to the $0 < u \leq 1$ domain of the Poincaré coordinates.

Coming back to the global coordinates, we introduce ρ by

$$\tan\rho = \sinh\chi \quad \text{with} \quad 0 \leq \rho < \frac{\pi}{2}, \tag{3.15}$$

so that (3.12) becomes

$$ds^2 = \frac{a^2}{\cos^2\rho}\left(-d\tau^2 + d\rho^2 + \sin^2\rho\, d\Omega_3^2\right). \tag{3.16}$$

The Penrose-diagrams of AdS_5 space-time and its universal covering space-time $CAdS_5$ are shown in Fig. 2, see [35, 36]. While the AdS_5 space-time contains closed timelike curves (denoted

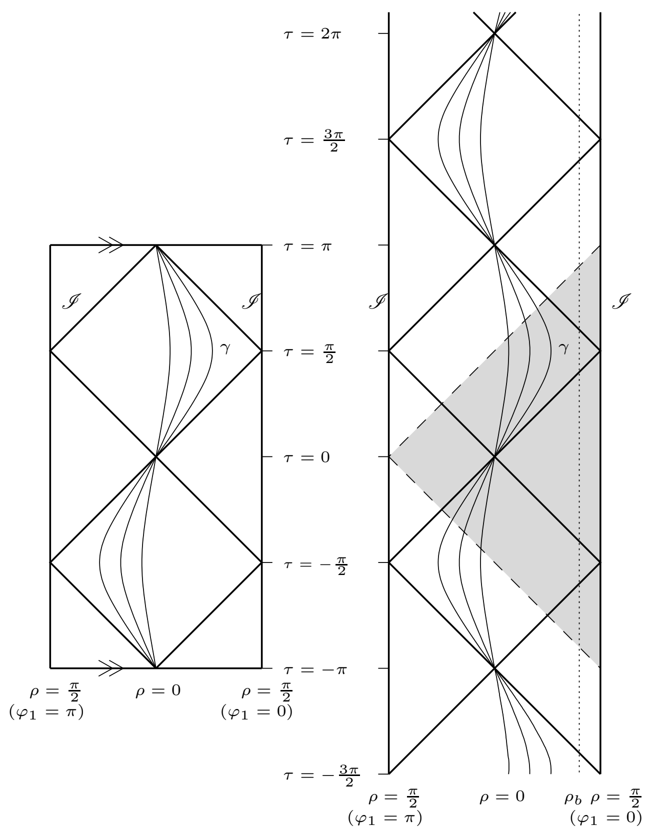

Figure 2: Penrose diagram of AdS_5 space and its universal covering space $CAdS_5$

by γ in the figure), its universal covering space-time $CAdS_5$ does not. The arrows in the left diagram indicate that the lines $\tau = -\pi$ and $\tau = \pi$ should be identified. The symbol $\mathscr{I}$ stands for the timelike surface $\rho = \pi/2$ (spatial infinity). It is this surface which is responsible for the absence of a Cauchy-surface in AdS-space. We will concentrate in the following on the $CAdS_5$ diagram. First we note that each point in the diagram corresponds to a 3-sphere. The shaded region indicates the patch covered by the Poincaré (and Randall-Sundrum) coordinates. Note that the position of the Randall-Sundrum brane cannot be represented in a simple way in the Penrose diagram of $CAdS_5$. The reason is that the u = const. hypersurfaces of the Poincaré coordinates generate a slicing of flat 4-dimensional Minkowski space-times while the points in the diagram represent (curved) 3-spheres. From the Penrose diagram it is immediately clear that the Randall-Sundrum-II space-time is geodesically incomplete. The timelike curves denoted by γ emanating from the origin ($\rho = 0, \tau = 0$) will all eventually exit the shaded region after a finite coordinate time τ. These geodesics appear inextendible from the point of view of the Randall-Sundrum space-time. The problems arise due to the arbitrary cutting of a space-time along the borders of a given coordinate patch which covers only a part of the initial space-time. Similar conclusions can be drawn for null geodesics, represented by lines making angles of 45 degrees with the vertical lines in Fig. 2. The vertical dotted line with topology $\mathbb{R} \times S^3$ at coordinate $\rho_b = \arctan\left[\sinh\left(\frac{\omega\pi}{2}\right)\right]$ close to $\rho = \pi/2$ in the right diagram corresponds to the location of our curved three brane, τ being proportional to our time coordinate t (see footnote 7). We note that since $\omega > 50$, this line is drawn by far too distant from $\rho = \pi/2$ as a simple expansion shows:

$$\rho_b = \arctan\left[\sinh\left(\frac{\omega\pi}{2}\right)\right] \sim \frac{\pi}{2} - 2e^{-\frac{\omega\pi}{2}} + \mathcal{O}(e^{-\frac{3\omega\pi}{2}})\,. \tag{3.17}$$

The slice of $CAdS_5$ to the right of ρ_b ($\rho_b \leq \rho \leq \pi/2$) is discarded and replaced by another copy of the slice to the left of ρ_b ($0 \leq \rho \leq \rho_b$).

From the Penrose diagram we can also deduce that our space-time is geodesically complete. Timelike geodesics (emanating from ($\rho = 0, \tau = 0$)) cross the brane ρ_b at some point or return to the origin depending on the initial velocities of the particles that define them. In both cases (as in the case of null geodesics) there is no obstacle to extend the affine parameter to larger and larger values.

As we already mentioned, the absence of a Cauchy surface in $CAdS$-space-time is due to the existence of the timelike surface $\mathscr{I}$ at spatial infinity. By construction, our space-time excludes $\mathscr{I}$ so that the natural question arises whether it is legitimate to revert the above argument and conclude the existence of a Cauchy surface in our model. Though interesting, we do not intend to elaborate this question any further in this paper.

4 Gravity localization on the brane

There are at least two equivalent ways of addressing the problem of the localization of gravity in brane-world scenarios. Both ways have their advantages and disadvantages. The first way is based on a detailed study of the Kaluza-Klein excitations of the graviton. After integrating out the extra dimension(s) one obtains an effective 4 dimensional Lagrangian involving the full tower of Kaluza-Klein gravitons. Considering the low energy scattering process of two test particles on the brane via exchange of Kaluza-Klein excitations allows to relate the non-relativistic scattering amplitude to the static potential felt by the two test particles. Since the coupling of each individual Kaluza-Klein particle to matter on the brane is proportional to the value of its transverse wave-function at the position of the brane, this approach necessitates a proper normalization of all Kaluza-Klein modes. An advantage of this approach is the possibility of distinguishing between the contributions to the potential coming from the zero mode (Newton's law) and from the higher modes (corrections thereof).

The second approach is based on a direct calculation of the graviton two-point function in the space-time under consideration, having the obvious advantage of bypassing all technicalities related to the Kaluza-Klein spectrum and the wave-function normalization. However, a physical interpretation of the effects of individual Kaluza-Klein modes from the point of view of a 4-dimensional observer is, to say the least, not straightforward. Finally, due to the equivalence of the two approaches it is clearly possible to fill this gap by reading of the Kaluza-Klein spectrum and the (normalized) wave-functions from the two-point function by locating its poles and determining corresponding residues.

In the setup considered in this paper we choose to work in the second approach, the direct evaluation of the Green's functions, due to additional difficulties arising from the non-Minkowskian nature of the induced metric on the brane. The lack of Poincaré invariance on the brane clearly implies the absence of this symmetry in the effective 4-dimensional Lagrangian as well as non-Minkowskian dispersion relations for the Kaluza-Klein modes.

4.1 Perturbation equations and junction conditions

We would like to study the fluctuations H_{ij} of the metric in the background (2.1) defined by:

$$ds^2 = -\sigma^2 dt^2 + R_U^2 \gamma^2 \left[\eta_{ij} + 2H_{ij}\right] d\varphi^i d\varphi^j + R^2 d\theta^2. \tag{4.1}$$

H_{ij} transform as a transverse, traceless second-rank tensor with respect to coordinate transformations on the maximally symmetric space with metric η_{ij}, a 3-sphere in our case:

$$\eta^{ij} H_{ij} = 0, \quad \eta^{ij} \tilde{\nabla}_i H_{jk} = 0, \tag{4.2}$$

where $\tilde{\nabla}$ denotes the covariant derivative associated with the metric η_{ij} on S^3. The gauge invariant symmetric tensor H_{ij} does not couple to the vector and scalar perturbations. Its perturbation equation in the bulk is obtained from the transverse traceless component of the perturbed Einstein equations in 5 dimensions:

$$\delta R_{MN} - \frac{2}{3}\Lambda \delta g_{MN} = 0\,. \tag{4.3}$$

Specifying this equation to the case of interest of a static background we find:

$$\frac{1}{\sigma^2}\ddot{H} - \frac{1}{R_U^2\gamma^2}\tilde{\Delta} H_{ij} - \frac{1}{R^2} H_{ij}'' - \frac{1}{R^2}\left(\frac{\sigma'}{\sigma} + 3\frac{\gamma'}{\gamma}\right) H_{ij}' + \frac{2}{R_U^2 \gamma^2} H_{ij} = 0, \tag{4.4}$$

where $\tilde{\Delta}$ denotes the Laplacian on S^3. The last equation is valid in the bulk in the absence of sources. We can now conveniently expand H_{ij} in the basis of the symmetric transverse traceless tensor harmonics $\hat{T}_{ij}^{(l\lambda)}$ on S^3 [37, 38, 39]:

$$H_{ij} = \sum_{l=2}^{\infty} \sum_{\lambda} \Phi^{(l\lambda)}(t,\theta) \hat{T}_{ij}^{(l\lambda)}, \tag{4.5}$$

where the $\hat{T}_{ij}^{(l\lambda)}$ satisfy

$$\begin{aligned} &\tilde{\Delta}\hat{T}_{ij}^{(l\lambda)} + k_l^2 \hat{T}_{ij}^{(l\lambda)} = 0, \qquad k_l^2 = l(l+2) - 2, \qquad l = 2, 3, ... \\ &\qquad \eta^{ij}\tilde{\nabla}_i \hat{T}_{jk}^{(l\lambda)} = 0, \qquad \eta^{ij}\hat{T}_{ij}^{(l\lambda)} = 0, \qquad \hat{T}_{[ij]}^{(l\lambda)} = 0. \end{aligned} \tag{4.6}$$

The sum over λ is symbolic and replaces all eigenvalues needed to describe the full degeneracy of the subspace of solutions for a given value of l. Introducing the expansion (4.5) into (4.4) and using the orthogonality relation of the tensor harmonics [38]

$$\int \sqrt{\eta}\, \eta^{ik}\, \eta^{jl}\, \hat{T}_{ij}^{(l\lambda)}\, \hat{T}_{kl}^{(l'\lambda')} d^3\varphi = \delta^{ll'}\, \delta^{\lambda\lambda'}\,, \tag{4.7}$$

we obtain:
$$\frac{1}{\sigma^2}\ddot{\Phi}^{(l\lambda)} - \frac{1}{R^2}\Phi^{(l\lambda)\prime\prime} - \frac{1}{R^2}\left(\frac{\sigma'}{\sigma} + 3\frac{\gamma'}{\gamma}\right)\Phi^{(l\lambda)\prime} + \frac{l(l+2)}{R_U^2\gamma^2}\Phi^{(l\lambda)} = 0. \tag{4.8}$$
This equation has to be compared to the equation of a massless scalar field in the background (2.1). After expanding the massless scalar in the corresponding scalar harmonics on S^3 we recover (4.8) with the only difference in the eigenvalue parameters due to different spectra of the Laplacian $\tilde{\Delta}$ for scalars and for tensors. Motivated by this last observation and in order to avoid the technicalities related to the tensorial nature of the graviton H_{ij} we confine ourselves in this paper to the study of the Green's functions of a massless scalar field in the background (2.1).

The differential equation (4.8) alone does not determine H_{ij} uniquely. We have to impose proper boundary conditions for H_{ij}. While imposing square integrability will constitute one boundary condition at $\theta = \pm\pi/2$, the behavior of H_{ij} on the brane will be dictated by the Israel junction condition [40]:
$$: K_{\mu\nu} : \ = -\frac{8\pi}{M^3}\left(T_{\mu\nu} - \frac{1}{3}T_\kappa{}^\kappa \bar{g}_{\mu\nu}\right). \tag{4.9}$$
Here $K_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ denote the extrinsic curvature and the induced metric on the brane, respectively. $T_{\mu\nu}$ is the 4-dimensional stress-energy tensor on the brane[8] and the symbol $: \ldots :$ is defined in (2.15). In our coordinate system the extrinsic curvature is given by:
$$K_{\mu\nu} = \frac{1}{2R}\frac{\partial \bar{g}_{\mu\nu}}{\partial\theta}. \tag{4.10}$$
Hence, the non-trivial components of the Israel condition become:
$$: \frac{\sigma'}{\sigma} : = -\frac{8\pi R}{M^3}\left(\frac{2}{3}c_0 - c\right), \tag{4.11}$$
$$: \frac{\gamma'}{\gamma}\left(\eta_{ij} + 2H_{ij}\right) + H'_{ij} : = \frac{8\pi R}{3M^3}c_0\left(\eta_{ij} + 2H_{ij}\right). \tag{4.12}$$
Separating the background from the fluctuation in (4.12) we find:
$$: \frac{\gamma'}{\gamma} : = \frac{8\pi R}{3M^3}c_0, \tag{4.13}$$
$$: H'_{ij} : = 0, \tag{4.14}$$
where we only used the continuity of H_{ij} on the brane. While eq. (4.13) directly coincides with eq. (2.12) of section (2.3), eq. (4.11) turns out to be a linear combination of (2.12) and (2.13).

Eq. (4.14) taken alone implies the continuity of H'_{ij} on the brane. If in addition the fluctuations are supposed to satisfy the $\theta \to -\theta$ symmetry, this condition reduces to a Neumann condition, as for example in our treatment of the scalar two-point function in the Randall-Sundrum II case (see appendix A).

In the next section we are going to find the static Green's functions of a massless scalar field in the Einstein static universe background. This will serve as a preparatory step for how to handle the more complicated case of a non-invertible Laplacian. More importantly, it will provide the necessary reference needed for the interpretation of the effect of the extra dimension on the potential between two test masses on the brane.

[8] $T^0{}_0 = c_0$, and $T^i{}_j = \delta^i{}_j c$ in the notation of section (2.3).

4.2 Gravity in the Einstein static universe

As announced in the previous section, we will concentrate on the massless scalar field. Our aim is to solve the analog of Poisson equation in the Einstein static universe background. Due to its topology $\mathbb{R} \times S^3$ we will encounter a difficulty related to the existence of a zero eigenvalue of the scalar Laplacian on S^3 necessitating the introduction of a modified Green's function.[9] Note that in the more physical case where the full tensor structure of the graviton is maintained no such step is necessary since all eigenvalues of the tensorial Laplacian are strictly negative on S^3, see (4.6).

4.2.1 Definition of a modified Green's function

We take the metric of Einstein's static universe in the form:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + A^2 d\Omega_3^2 \tag{4.15}$$

with $d\Omega_3^2$ being the line element of a 3-sphere given in (2.2) and A being its constant radius. The equation of a massless scalar field then becomes:

$$\frac{1}{\sqrt{-g}} \partial_\mu \left[\sqrt{-g}\, g^{\mu\nu} \partial_\nu u(t, \vec{x}) \right] = j(t, \vec{x}) \tag{4.16}$$

which in the static case reduces to

$$\mathcal{D} u(\vec{x}) \equiv \frac{1}{A^2} \tilde{\Delta} u(\vec{x}) = j(\vec{x}) \,, \tag{4.17}$$

where $\tilde{\Delta}$ is the scalar Laplacian on S^3 and we choose to write $\vec{x}$ for the collection of the three angles on S^3. All functions involved are supposed to obey periodic boundary condition on S^3 so that in Green's identity all boundary terms vanish:

$$\int \sqrt{-g}\; (\mathcal{D}\, v(\vec{x}))\; \overline{u}(\vec{x})\, d^3\vec{x} = \int \sqrt{-g}\, v(\vec{x})\, \overline{\mathcal{D}\, u(\vec{x})}\, d^3\vec{x} = \int \sqrt{-g}\, v(\vec{x})\, \overline{j}(\vec{x})\, d^3\vec{x} \,. \tag{4.18}$$

Here and in the following bars denote complex conjugate quantities. The operator $\mathcal{D}$ trivially allows for an eigenfunction with zero eigenvalue (the constant function on S^3):

$$\mathcal{D} u_0(\vec{x}) = 0 \,, \qquad \int \sqrt{-g}\, u_0(\vec{x})\, \overline{u}_0(\vec{x})\, d^3\vec{x} = 1 \,. \tag{4.19}$$

Then, since the homogeneous equation has not only the trivial solution ($u = 0$) which satisfies the periodic boundary conditions in the angular variables, the operator $\mathcal{D}$ cannot be invertible. Therefore, there is no solution to the equation (4.17) for an arbitrary source. In order to still define a "modified" Green's function we have to restrict the possible sources to sources that satisfy the following solvability condition:

$$\int \sqrt{-g}\, j(\vec{x})\, \bar{u}_0(\vec{x})\, d^3\vec{x} = 0. \tag{4.20}$$

Formally this condition can be obtained by replacing $v(\vec{x})$ by the non-trivial solution of the homogeneous equation $u_0(\vec{x})$ in (4.18). We now define the modified Green's function as a solution of

$$\mathcal{D}_x \mathcal{G}(\vec{x}, \vec{x}') = \frac{\delta^3(\vec{x} - \vec{x}')}{\sqrt{-g}} - u_0(\vec{x})\, \bar{u}_0(\vec{x}') \,. \tag{4.21}$$

[9] For an elementary introduction to the concept of modified Green's functions see e.g [41].

Replacing now $v(\vec{x})$ by $\mathcal{G}(\vec{x},\vec{x}')$ in Green's identity (4.18) we obtain (after complex conjugation) the desired integral representation for the solution of (4.17):

$$u(\vec{x}) = C u_0(\vec{x}) + \int \sqrt{-g}\, j(\vec{x}')\, \overline{\mathcal{G}}(\vec{x}',\vec{x})\, d^3\vec{x}' \,, \tag{4.22}$$

where C is given by

$$C = \int \sqrt{-g}\, u(\vec{x}')\, \overline{u}_0(\vec{x}')\, d^3\vec{x}' \,. \tag{4.23}$$

Note that the source

$$j_0(\vec{x}) = \frac{\delta^3(\vec{x}-\vec{x}')}{\sqrt{-g}} - u_0(\vec{x})\bar{u}_0(\vec{x}') \tag{4.24}$$

trivially satisfies the solvability condition (4.20) and represents a point source at the location $\vec{x}=\vec{x}'$ compensated by a uniform negative mass density. To fix the normalization of $u_0(\vec{x})$ we write:

$$u_0(\vec{x}) = N_0 \Phi_{100}(\vec{x}) \;\text{ with }\; \int \sqrt{\eta}\, \Phi_{100}(\vec{x}) \overline{\Phi}_{100}(\vec{x}) d^3\vec{x} = 1, \tag{4.25}$$

where $\Phi_{100} = \frac{1}{\pi\sqrt{2}}$ and so

$$\int \sqrt{-g}\, u_0(\vec{x})\, \overline{u}_0(\vec{x})\, d^3\vec{x} = |N_0|^2 A^3 \underbrace{\int \sqrt{\eta}\, \Phi_{100}(\vec{x})\, \overline{\Phi}_{100}(\vec{x})\, d^3\vec{x}}_{=1} = |N_0|^2 A^3 = 1 \,, \tag{4.26}$$

so that

$$u_0(\vec{x}) = \frac{1}{A^{3/2}} \Phi_{100}(\vec{x}). \tag{4.27}$$

In order to solve the differential equation (4.21) defining the modified Green's function $\mathcal{G}(\vec{x},\vec{x}')$, we expand in eigenfunctions of the Laplace operator on S^3, the so-called scalar harmonics $\Phi_{\lambda lm}$ with properties (see e.g. [42, 43]):

$$\tilde{\Delta}\Phi_{\lambda lm} = (1-\lambda^2)\Phi_{\lambda lm} \qquad \lambda = 1,2,\ldots; l = 0,\ldots,\lambda-1; m=-l,\ldots,l \,. \tag{4.28}$$

Our ansatz therefore reads:

$$\mathcal{G}(\vec{x},\vec{x}') = \sum_{\lambda=1}^{\infty}\sum_{l=0}^{\lambda-1}\sum_{m=-l}^{l} \Phi_{\lambda lm}(\vec{x})\, c_{\lambda lm}(\vec{x}') \,. \tag{4.29}$$

Introducing this in (4.21) we obtain

$$\sum_{\lambda=1}^{\infty}\sum_{l=0}^{\lambda-1}\sum_{m=-l}^{l} \left[-\frac{\lambda^2-1}{A^2} c_{\lambda lm}(\vec{x}') \right] \Phi_{\lambda lm}(\vec{x}) = \frac{\delta^3(\vec{x}-\vec{x}')}{\sqrt{-g}} - u_0(\vec{x})\, \overline{u}_0(\vec{x}') \,. \tag{4.30}$$

If we now multiply by $\Phi_{\lambda' l' m'}(\vec{x})\sqrt{\eta}$ and integrate over S^3 we find

$$-\frac{\lambda'^2-1}{A^2} c_{\lambda' l' m'}(\vec{x}') = \frac{1}{A^3} \overline{\Phi}_{\lambda' l' m'}(\vec{x}') - \bar{u}_0(\vec{x}') \int_{S^3} \sqrt{\eta}\, u_0(\vec{x})\, \overline{\Phi}_{\lambda' l' m'}(\vec{x})\, d^3\vec{x}, \tag{4.31}$$

where we made use of the orthogonality relation of the scalar harmonics

$$\int \sqrt{\eta}\, \overline{\Phi}_{\lambda lm}(\vec{x})\, \Phi_{\lambda' l' m'}(\vec{x})\, d^3\vec{x} = \delta_{\lambda\lambda'}\delta_{ll'}\delta_{mm'} . \tag{4.32}$$

In the case $\lambda' = 1$ (and vanishing l' and m') the above equation is identically satisfied for all values of $c_{100}(\vec{x}')$.[10]

In the case $\lambda' \neq 1$ the coefficient $c_{\lambda' l' m'}(\vec{x}')$ follows to be

$$c_{\lambda' l' m'}(\vec{x}') = \frac{\overline{\Phi}_{\lambda' l' m'}(\vec{x}')}{A(1-\lambda'^2)} \tag{4.33}$$

so that the formal solution for the modified Green's function can be written as

$$\mathcal{G}(\vec{x}, \vec{x}') = \sum_{\lambda=2}^{\infty} \sum_{l=0}^{\lambda-1} \sum_{m=-l}^{l} \frac{\Phi_{\lambda lm}(\vec{x})\bar{\Phi}_{\lambda lm}(\vec{x}')}{A(1-\lambda^2)} . \tag{4.34}$$

Due to the maximal symmetry of the 3-sphere, the Green's function $\mathcal{G}(\vec{x}, \vec{x}')$ can only depend on the geodesic distance $s(\vec{x}, \vec{x}') \in [0, \pi]$ between the two points $\vec{x}$ and $\vec{x}'$:

$$\begin{aligned} \cos s &= \cos\varphi_1 \cos\varphi_1' + \sin\varphi_1 \sin\varphi_1' \cos\beta , \\ \cos\beta &= \cos\varphi_2 \cos\varphi_2' + \sin\varphi_2 \sin\varphi_2' \cos(\varphi_3 - \varphi_3') , \end{aligned} \tag{4.35}$$

which in the case $\varphi_2 = \varphi_2'$ and $\varphi_3 = \varphi_3'$ clearly reduces to $s = \varphi_1 - \varphi_1'$. Indeed, the sum over l and m can be performed using[11]

$$\sum_{l=0}^{\lambda-1} \sum_{m=-l}^{l} \bar{\Phi}_{\lambda lm}(\vec{x})\Phi_{\lambda lm}(\vec{x}') = \frac{\lambda}{2\pi^2} \frac{\sin(\lambda s)}{\sin s} \tag{4.36}$$

such that even the remaining sum over λ can be done analytically:

$$\tilde{\mathcal{G}}(s) \equiv \mathcal{G}(\vec{x}, \vec{x}') = \frac{1}{8\pi^2 A} - \frac{1}{4\pi A}\left[(1 - \frac{s}{\pi}) \cot s\right] , \qquad s \in [0, \pi] . \tag{4.37}$$

To interpret this result, we develop $\tilde{\mathcal{G}}(s)$ around $s = 0$ obtaining[12]

$$\tilde{\mathcal{G}}(s) = \frac{1}{A}\left[-\frac{1}{4\pi s} + \frac{3}{8\pi^2} + \frac{s}{12\pi} + \mathcal{O}(s^2)\right] . \tag{4.38}$$

By introducing the variable $r = sA$ and by treating $\tilde{\mathcal{G}}(s)$ as a gravitational potential we find

$$\frac{1}{A}\frac{d\tilde{\mathcal{G}}(s)}{ds} = \frac{1}{4\pi r^2} + \frac{1}{12\pi A^2} + \mathcal{O}(r/A^3). \tag{4.39}$$

We notice that for short distances, $r \ll A$, we find the expected flat result whereas the corrections to Newton's law become important at distances r of the order of A in the form of a constant attracting force.[13]

[10] This means that we have the freedom to choose $c_{100}(\vec{x}')$ freely. Our choice is $c_{100}(\vec{x}') = 0$ without restricting generality since from eq. (4.22) we immediately conclude that for sources satisfying (4.20) there will never be any contribution to $u(\vec{x})$ coming from $c_{100}(\vec{x}')$.

[11] See e.g. [43].

[12] Note that the constant term in the expansion (4.38) is specific to our choice of $c_{100}(\vec{x}')$.

[13] There exists numerous articles treating the gravitational potential of a point source in Einstein's static universe (see [44, 45] and references therein). We only would like to point out here similarities between our result (4.37) and the line element of a Schwarzschild metric in an Einstein static universe background given in [44].

4.3 Modified Green's function of a massless scalar field in the background space-time (2.1)

We are now prepared to address the main problem of this work namely the computation of the modified Green's function of a massless scalar field in the background space-time (2.1). Since our main interest focuses again on the low energy properties of the two-point function, we will limit ourselves to the static case. The main line of reasoning is the same as in the previous section. Due to the fact that our space has the global topology of a 4-sphere S^4, we again are confronted with a non-invertible differential operator. We would like to solve

$$\mathcal{D}u(\varphi_i,\theta) \equiv \frac{\tilde{\Delta}u(\varphi_i,\theta)}{R_U^2\gamma(\theta)^2} + \frac{1}{R^2}\frac{1}{\sigma(\theta)\gamma(\theta)^3}\frac{\partial}{\partial\theta}\left[\sigma(\theta)\gamma(\theta)^3\frac{\partial u(\varphi_i,\theta)}{\partial\theta}\right] = j(\varphi_i,\theta)\,, \tag{4.40}$$

where the independent angular variable ranges are $0 \leq \varphi_1 \leq \pi$; $0 \leq \varphi_2 \leq \pi$; $0 \leq \varphi_3 \leq 2\pi$; $-\pi/2 \leq \theta \leq \pi/2$. Every discussion of Green's functions is based on Green's identity relating the differential operator under consideration to its adjoint operator. Since $\mathcal{D}$ is formally self-adjoint we have

$$\int_0^\pi d\varphi_1 \int_0^\pi d\varphi_2 \int_0^{2\pi} d\varphi_3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\sqrt{-g}\left[(\mathcal{D}v)\overline{u} - v\overline{(\mathcal{D}u)}\right] =$$

$$\int_0^\pi d\varphi_1 \int_0^\pi d\varphi_2 \int_0^{2\pi} d\varphi_3 \frac{R_U^3}{R}\sqrt{\eta}\left[\sigma(\theta)\gamma(\theta)^3\left(\bar{u}\frac{\partial v}{\partial\theta} - \frac{\partial\bar{u}}{\partial\theta}v\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \ldots\,, \tag{4.41}$$

where we dropped the arguments of u and v for simplicity. The dots in (4.41) refer to boundary terms in the variables φ_i and since we again employ an eigenfunction expansion in scalar harmonics on S^3, these boundary terms will vanish. In order to find an integral representation of the solution $u(\varphi_i,\theta)$ of (4.40) we have to impose appropriate boundary conditions on u and v at $\theta = \pm\pi/2$. For the time being we assume this to be the case such that all boundary terms in (4.41) vanish and proceed with the formal solution of (4.40). We will address the issue of boundary conditions in θ in detail in appendices A and B.

In the following we will collectively use x instead of (φ_i,θ). The homogeneous equation $\mathcal{D}u(x) = 0$ does not have a unique solution under the assumption of periodic boundary conditions. In addition to the trivial solution ($u = 0$) we also find

$$\mathcal{D}\,u_0(x) = 0\,, \qquad \int \sqrt{-g}\,u_0(x)\,\overline{u}_0(x)\,d^4x = 1\,. \tag{4.42}$$

Therefore, the corresponding inhomogeneous equation (4.40) does not have a solution unless we again restrict the space of allowed sources:

$$\int \sqrt{-g}\,j(x)\,\overline{u}_0(x)\,d^4x = 0. \tag{4.43}$$

As in the last section, this condition can be obtained by replacing v by the non-trivial solution of the homogeneous equation u_0 in (4.41). We now define the modified Green's function by

$$\mathcal{D}_x\mathcal{G}(x,x') = \frac{\delta^4(x-x')}{\sqrt{-g}} - u_0(x)\overline{u}_0(x')\,. \tag{4.44}$$

From Green's identity (4.41), with $v(x)$ given by $\mathcal{G}(x,x')$, we again obtain after complex conjugation the desired integral representation:

$$u(x) = Cu_0(x) + \int \sqrt{-g}\, j(x')\overline{\mathcal{G}(x',x)}\, d^4x' \,, \tag{4.45}$$

with

$$C = \int \sqrt{-g}\, u(x')\, \overline{u}_0(x')\, d^4x' \,. \tag{4.46}$$

As before, the source

$$j_0(x) = \frac{\delta^4(x-x')}{\sqrt{-g}} - u_0(x)\, \overline{u}_0(x') \tag{4.47}$$

satisfies the solvability condition (4.43) by construction. The normalization of the constant mode $u_0(x)$ is slightly more involved than before due to the nontrivial measure $\sigma(\theta)\gamma(\theta)^3$ in the θ integration. By inserting

$$u_0(x) = N_0\Phi_{100}(\varphi_i)\chi_1(\theta) \text{ with } \chi_1(\theta) = 1 \tag{4.48}$$

in the integral in (4.42) we obtain

$$N_0 = \left[\frac{2\omega}{RR_U^3 \tanh\left(\frac{\omega\pi}{2}\right)}\right]^{\frac{1}{2}} \,. \tag{4.49}$$

For the solution of eq. (4.44) we use the ansatz

$$\mathcal{G}(\varphi_i,\varphi_i',\theta,\theta') = \sum_{\lambda=1}^{\infty}\sum_{l=0}^{\lambda-1}\sum_{m=-l}^{l} \Phi_{\lambda lm}(\varphi_i)\, c_{\lambda lm}(\varphi_i',\theta,\theta') \,, \tag{4.50}$$

where from now on we decide to write all arguments explicitly. After inserting this in (4.44) we find

$$\sum_{\lambda=1}^{\infty}\sum_{l=0}^{\lambda-1}\sum_{m=-l}^{l} \left[-\frac{\lambda^2-1}{R_U^2\gamma(\theta)^2}c_{\lambda lm}(\varphi_i',\theta,\theta') + \frac{1}{R^2}\frac{1}{\sigma(\theta)\gamma(\theta)^3}\frac{\partial}{\partial\theta}\left(\sigma(\theta)\gamma(\theta)^3\frac{\partial c_{\lambda lm}(\varphi_i',\theta,\theta')}{\partial\theta}\right)\right]\Phi_{\lambda lm}(\varphi_i)$$
$$= \frac{\delta^3(\varphi_i-\varphi_i')\,\delta(\theta-\theta')}{\sqrt{-g}} - u_0(\varphi_i,\theta)\,\overline{u}_0(\varphi_i',\theta'). \tag{4.51}$$

After multiplication by $\sqrt{\eta}\,\overline{\Phi}_{\lambda' l' m'}(\varphi_i)$ and integration over S^3 we obtain

$$-\frac{\lambda'^2-1}{R_U^2\gamma(\theta)^2}c_{\lambda' l' m'}(\varphi_i',\theta,\theta') + \frac{1}{R^2}\frac{1}{\sigma(\theta)\gamma(\theta)^3}\frac{\partial}{\partial\theta}\left[\sigma(\theta)\gamma(\theta)^3\frac{\partial c_{\lambda' l' m'}(\varphi_i',\theta,\theta')}{\partial\theta}\right] \tag{4.52}$$
$$= \frac{1}{R_U^3 R}\frac{1}{\sigma(\theta)\gamma(\theta)^3}\delta(\theta-\theta')\overline{\Phi}_{\lambda' l' m'}(\varphi_i') - \overline{u}_0(\varphi_i',\theta')\int_{S^3}\sqrt{\eta}\, u_0(\varphi_i,\theta)\,\overline{\Phi}_{\lambda' l' m'}(\varphi_i)\, d^3\varphi_i \,.$$

We now have to distinguish the cases $\lambda'=1$ and $\lambda'\neq 1$.

1. $(\lambda' l' m') = (100)$.

   In this case, the last term on the right hand side of (4.52) will give a non-vanishing contribution:

$$\frac{1}{R^2}\frac{1}{\sigma(\theta)\gamma(\theta)^3}\frac{\partial}{\partial\theta}\left[\sigma(\theta)\gamma(\theta)^3\frac{\partial c_{100}(\varphi_i',\theta,\theta')}{\partial\theta}\right] = \frac{1}{R_U^3 R}\frac{1}{\sigma(\theta)\gamma(\theta)^3}\delta(\theta-\theta')\overline{\Phi}_{100}(\varphi_i') - \quad (4.53)$$
$$|N_0|^2\bar{\Phi}_{100}(\varphi_i')\chi_1(\theta)\overline{\chi}_1(\theta')\underbrace{\int_{S^3}\sqrt{\eta}\,\Phi_{100}(\varphi_i)\overline{\Phi}_{100}(\varphi_i)\,d^3\varphi_i}_{=1}\,.$$

   By defining $g^{(1)}(\theta,\theta')$ by the relation

$$c_{100}(\varphi_i',\theta,\theta') = \frac{1}{R_U^3 R}\bar{\Phi}_{100}(\varphi_i')g^{(1)}(\theta,\theta') \quad (4.54)$$

   we obtain the following differential equation for $g^{(1)}(\theta,\theta')$:

$$\frac{1}{R^2}\frac{1}{\sigma(\theta)\gamma(\theta)^3}\frac{\partial}{\partial\theta}\left[\sigma(\theta)\gamma(\theta)^3\frac{\partial g^{(1)}(\theta,\theta')}{\partial\theta}\right] = \frac{1}{\sigma(\theta)\gamma(\theta)^3}\delta(\theta-\theta') - \tilde{\chi}_1(\theta)\bar{\tilde{\chi}}_1(\theta')\,, \quad (4.55)$$

   where we used

$$\tilde{\chi}_1(\theta) \equiv \left[\frac{2\omega}{\tanh\left(\frac{\omega\pi}{2}\right)}\right]^{\frac{1}{2}}\chi_1(\theta)\,, \qquad (\chi_1(\theta)\equiv 1)\,. \quad (4.56)$$

   Note that we defined $\tilde{\chi}_1(\theta)$ in such a way that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sigma(\theta)\gamma(\theta)^3\tilde{\chi}_1(\theta)\bar{\tilde{\chi}}_1(\theta)d\theta = 1. \quad (4.57)$$

2. $(\lambda' l' m') \neq (100)$.

   Due to the orthogonality of Φ_{100} and $\Phi_{\lambda' l' m'}$ on S^3, the last term on the right hand side of (4.52) vanishes. We therefore have

$$-\frac{\lambda'^2-1}{R_U^2\gamma(\theta)^2}c_{\lambda' l' m'}(\varphi_i',\theta,\theta') + \frac{1}{R^2}\frac{1}{\sigma(\theta)\gamma(\theta)^3}\frac{\partial}{\partial\theta}\left[\sigma(\theta)\gamma(\theta)^3\frac{\partial c_{\lambda' l' m'}(\varphi_i',\theta,\theta')}{\partial\theta}\right]$$
$$= \frac{1}{R_U^3 R}\frac{1}{\sigma(\theta)\gamma(\theta)^3}\delta(\theta-\theta')\bar{\Phi}_{\lambda' l' m'}(\varphi_i')\,. \quad (4.58)$$

   Introducing $g^{(\lambda')}(\theta,\theta')$ again via

$$c_{\lambda' l' m'}(\varphi_i',\theta,\theta') = \frac{1}{R_U^3 R}\bar{\Phi}_{\lambda' l' m'}(\varphi_i')g^{(\lambda')}(\theta,\theta')\,, \quad (4.59)$$

   we see that $g^{(\lambda')}(\theta,\theta')$ has to satisfy

$$\frac{1-\lambda'^2}{R_U^2\gamma(\theta)^2}g^{(\lambda')}(\theta,\theta') + \frac{1}{R^2}\frac{1}{\sigma(\theta)\gamma(\theta)^3}\frac{\partial}{\partial\theta}\left[\sigma(\theta)\gamma(\theta)^3\frac{\partial g^{(\lambda')}(\theta,\theta')}{\partial\theta}\right] = \frac{\delta(\theta-\theta')}{\sigma(\theta)\gamma(\theta)^3}\,. \quad (4.60)$$

Combining the above results for $\lambda = 1$ and $\lambda \neq 1$ we are able to write the formal solution of (4.44) as

$$\mathcal{G}(\varphi_i, \varphi_i', \theta, \theta') = \frac{1}{R_U^3 R} \sum_{\lambda=1}^{\infty} \sum_{l=0}^{\lambda-1} \sum_{m=-l}^{l} \Phi_{\lambda lm}(\varphi_i) \bar{\Phi}_{\lambda lm}(\varphi_i') g^{(\lambda)}(\theta, \theta') \,. \tag{4.61}$$

We obtain a further simplification of this formal solution by employing the spherical symmetry on S^3, see eq. (4.36), leaving us with a representation of the two-point function by a Fourier sum, which is natural for a compact space without boundaries:

$$\tilde{\mathcal{G}}(s, \theta, \theta') = \mathcal{G}(\varphi_i, \varphi_i', \theta, \theta') = \frac{1}{R R_U^3} \sum_{\lambda=1}^{\infty} \frac{\lambda}{2\pi^2} \frac{\sin(\lambda s)}{\sin s} g^{(\lambda)}(\theta, \theta') \,. \tag{4.62}$$

Finding the formal solution (4.62) was straightforward apart from minor complications inherent to the use of a modified Green's function. To solve the differential equations (4.55) and (4.60) by imposing appropriate boundary conditions again is a routine task without any conceptual difficulties, though slightly technical in nature. Replacing back this solution in (4.62) we are left with a Fourier sum which at first sight looks intractable, anticipating the fact that the solutions $g^{(\lambda)}(\theta, \theta')$ (for $\lambda \neq 1$) are given by hypergeometric functions. Nevertheless it is possible to extract the desired asymptotic information from the sum (4.62).

In order not to disturb the transparency and fluidity of the main article we provide large parts of the technical calculations in four appendices. In appendix A we report similarities and differences in the evaluation of the two-point functions between our case and the case of Randall-Sundrum, since the latter served as a guideline for handling the more difficult case under consideration. The solutions of eqs. (4.55) and (4.60) are presented in appendix B and the evaluation of the Fourier sum (4.62) at distances exceeding the size of the extra dimension in appendix C. Eventually, appendix D contains the evaluation of the Fourier sum (4.62) for distances smaller than the extra dimension. In this way, we can offer the reader less interested in the details of the computations to have the main results at hand.

From its definition (4.44) we understand that the Green's function (4.62) can be considered as the response of the scalar field to the combination of a point-like source located at coordinates (φ_i', θ') and a delocalized, compensating negative contribution. Since we would like to see the response to a point-like particle on the brane we put $\theta' = 0$ in eq. (4.62) and explicitly write the $\lambda = 1$ term:

$$\tilde{\mathcal{G}}(s, \theta, 0) = \frac{g^{(1)}(\theta, 0)}{2\pi^2 R R_U^3} + \frac{R}{4\pi^2 R_U^3} \frac{1}{\sin s} S[s, \theta, \omega] \,, \tag{4.63}$$

with $S[s, \theta, \omega]$ given by (C.1) of appendix C (see also (B.13) of appendix B). The general result for the sum $S[s, 0, \omega]$ obtained in appendix C is

$$\begin{aligned} S[s, 0, \omega] \equiv \lim_{\theta \to 0} S[s, \theta, \omega] = &-\frac{2}{\omega} \frac{z(0)^{\frac{1}{2}}}{1 - z(0)} \left(\frac{\pi - s}{2} \cos s - \frac{1}{4} \sin s \right) \\ &- \frac{1}{2\omega} z(0)^{-\frac{1}{2}} \ln\left[1 - z(0)\right] \sin s - \frac{1}{\omega} z(0)^{\frac{1}{2}} \lim_{\theta \to 0} R[s, \theta, \omega] \,, \end{aligned} \tag{4.64}$$

where we have $z(0) = \tanh^2\left(\frac{\omega\pi}{2}\right)$ and where we refer to (C.9) for the definition of $R[s, \theta, \omega]$. The first term in (4.64) is the zero mode contribution[14] and we see that it reproduces exactly the 4-dimensional static Green's function of Einstein's static universe given in (4.37). The other two terms are the contributions from the higher Kaluza-Klein modes.

We first concentrate on the case where $s \sim 1$ or what is equivalent $r \sim R_U$. Since one can easily convince oneself that $R[s,0,\omega] \sim [1-z(0)]^0$ in this regime, the contributions of the higher Kaluza-Klein modes are strongly suppressed with respect to the zero mode contribution. This means that as in the case of the Randall-Sundrum-II model it is the zero mode which dominates the behavior of gravity at distances much larger than the extra dimensions R. The main difference to the Randall-Sundrum-II case is that the zero mode of our model not only gives rise to the typical 4-dimensional $1/r$ singularity but also accounts for the compactness of space by reproducing the Einstein static universe behavior (4.37). This result is somewhat surprising given the extreme anisotropy of our manifold. Due to the fact that the distances between two arbitrary points on the brane are of the order of R, one might intuitively expect that the extra dimension can be effective in determining also the large distance behavior of gravity (on the brane). As we could show by direct calculation the above expectation turns out to be incorrect.

Next we consider physical distances r much larger than the extra dimension $r \gg R$ and much smaller than the observable universe $r \ll R_U$ in which case the results for $R[s,0,\omega]$ can be seen to be:

$$R[s,0,\omega] \sim \frac{\pi}{2s^2} + \frac{\pi}{2}\frac{1-z(0)}{s^4}\left\{8 - 6\ln 2 - 6\ln\left[s\,[1-z(0)]^{-1/2}\right]\right\} + \mathcal{O}\left[[1-z(0)]^2\,\frac{\ln\left[s\,[1-z(0)]^{-1/2}\right]}{s^6}\right] \tag{4.65}$$

valid for $[1-z(0)]^{1/2} \ll s \ll 1$. The zero mode contribution (to $S[s,0,\omega]$) in this regime is simply the constant obtained by setting $s=0$ in the first term of (4.64) so that we obtain:

$$S[s,0,\omega] \sim -\frac{\pi}{\omega}\frac{z(0)^{\frac{1}{2}}}{1-z(0)}\left\{1 + \frac{1}{2\bar{s}^2} + \frac{1}{\bar{s}^4}\left[4 - 3\ln 2 - 3\ln\bar{s}\right] + \mathcal{O}\left(\frac{\ln\bar{s}}{\bar{s}^6}\right)\right\}, \tag{4.66}$$

where we introduced $\bar{s} = s\,[1-z(0)]^{-1/2}$. Inserting this result in (4.63) and using physical distance $r = R_U s$ instead of s we obtain

$$\begin{aligned}\tilde{\mathcal{G}}(s,0,0) &= \frac{g^{(1)}(0,0)}{2\pi^2 R R_U^3} - \frac{1}{4\pi r}\frac{\omega}{R}\coth\left(\frac{\omega\pi}{2}\right)\left\{1 + \frac{1}{2\bar{s}^2} + \frac{1}{\bar{s}^4}\left[4 - 3\ln 2 - 3\ln\bar{s}\right] + \mathcal{O}\left(\frac{\ln\bar{s}}{\bar{s}^6}\right)\right\} = \\ &= \frac{g^{(1)}(0,0)}{2\pi^2 R R_U^3} - \frac{1}{4\pi r}\frac{\omega}{R}\coth\left(\frac{\omega\pi}{2}\right)\left\{1 + \frac{\tanh^2\left(\frac{\omega\pi}{2}\right)}{2\bar{r}^2} + \frac{\tanh^4\left(\frac{\omega\pi}{2}\right)}{\bar{r}^4}\times\right. \\ &\left.\times\left[4 - 3\ln 2 - 3\ln\left(\frac{\bar{r}}{\tanh\left(\frac{\omega\pi}{2}\right)}\right)\right] + \mathcal{O}\left[\frac{\tanh^6\left(\frac{\omega\pi}{2}\right)}{\bar{r}^6}\ln\left(\frac{\bar{r}}{\tanh\left(\frac{\omega\pi}{2}\right)}\right)\right]\right\}.\end{aligned} \tag{4.67}$$

[14] At this point we have to explain what we mean by "zero mode" in this context, since due to the asymmetrically warped geometry, the spectrum of Kaluza-Klein excitations will not be Lorentz-invariant and strictly speaking, different excitations cannot be characterized by different 4-dimensional masses. More accurate would be to say that for a given value of the momentum eigenvalue λ there exists a tower of corresponding Kaluza-Klein excitations with energies given by E_n^λ with $n = 0, 1, \ldots$. In our use of language the "zero mode branch" of the spectrum or simply the "zero mode" is defined to be the collection of the lowest energy excitations corresponding to all possible values of λ, that is by the set $\left\{\left(E_0^\lambda, \lambda\right), \lambda = 1, 2, \ldots\right\}$. The definition is readily generalized to higher branches of the spectrum.

Since we would like to compare our result with the corresponding correction in the Randall-Sundrum II case, we introduced the dimensionless distance variable $\bar{r} = r\omega/R$, the physical distance measured in units of the AdS-radius, in the last line of the above result. We see that in complete agreement with the Randall-Sundrum II scenario, our setup reproduces 4-dimensional gravity at large distances with extremely suppressed corrections. The only remnant effect from the different global topology manifest itself through the factors of $\tanh\left(\frac{\omega\pi}{2}\right)$ and $\coth\left(\frac{\omega\pi}{2}\right)$ which are very close to 1. We furthermore emphasize that apart from these deviations, the asymptotic we obtained coincides exactly with the asymptotic for the case of a massless scalar field in the Randall-Sundrum-II background, see (A.20) and e.g. [46, 47, 48, 49]. From the factor $\omega\coth\left(\frac{\omega\pi}{2}\right)/R$ in eq. (4.67) we see that also the relation between the fundamental scale M and the Planck-scale M_{Pl} gets modified only by the same factor of $\tanh\left(\frac{\omega\pi}{2}\right)$:

$$M_{Pl}^2 = M^3 \frac{R}{\omega} \tanh\left(\frac{\omega\pi}{2}\right), \tag{4.68}$$

where we remind that R/ω is nothing but the AdS-Radius.

Eventually, we treat the case of distances inferior to the extra dimension $r \ll R$. After using the result (D.12) of appendix D in (4.63), a short calculation reveals

$$\tilde{\mathcal{G}}(s,0,0) = \frac{g^{(1)}(0,0)}{2\pi^2 R R_U^3} - \frac{1}{4\pi^2}\frac{1}{R_U^2 s^2 + R^2\theta^2}, \tag{4.69}$$

a result that has to be compared to the characteristic solution of the Poisson equation in 4-dimensional flat space. In n-dimensional flat space one has:

$$\Delta\left[-\frac{1}{(n-2)V_{S^{n-1}} r^{n-2}}\right] = \delta(r)\,, \qquad r = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \tag{4.70}$$

with $V_{S^{n-1}} = 2\pi^{\frac{n}{2}}/\Gamma[\frac{n}{2}]$ denoting the volume of the $n-1$ sphere. Specifying to $n = 4$, we recover the correct prefactor of $-1/4\pi^2$ in (4.69) multiplying the $1/r^2$ singularity.

Finally, we mention that in none of the considered cases we payed any attention to the additive constant in the two-point function on the brane. The arbitrary constant entering the solution $g^{(1)}(\theta,\theta')$ can always be chosen in such a way that $g^{(1)}(0,0)$ vanishes on the brane (see appendix B).

5 Conclusions

In this paper we considered a particular brane world model in 5 dimensions with the characteristic property that the spatial part of the space-time manifold (including the extra dimension) is compact and has the topology of a 4-sphere S^4. Similar to the original Randall-Sundrum II model, the 3-brane is located at the boundary between two regions of AdS_5 space-time. The coordinates of AdS_5 used by Randall and Sundrum are closely related to the so-called Poincaré coordinates of AdS_5. While the extra dimension in this set of coordinates provides a slicing of AdS_5 along flat 4-dimensional Minkowski sections (resulting in a flat Minkowskian induced metric on the brane), their disadvantage is that they do not cover the whole of AdS_5 space-time.

The coordinates we used in this paper are the global coordinates of AdS_5 known to provide a global cover of the AdS_5 space-time. In this case the “extra” dimension labels different sections with intrinsic geometry $\mathbb{R} \times S^3$, the geometry of Einsteins static universe. The induced metric on the 3-brane in our setup is therefore also given by $\mathbb{R} \times S^3$.

As we illustrated with the use of the Penrose-diagram of AdS_5, the incompleteness of the Poincaré patch is at the origin of the incompleteness of the Randall Sundrum II space-time with respect to timelike and lightlike geodesics. Moreover we were able to demonstrate that the setup considered in this paper provides an alternative to the Randall-Sundrum II model which does not suffer from the drawback of being geodesically incomplete. The latter point was part of the main motivations for this work.

The spatial part of our manifold is characterized by an extreme anisotropy with respect to one of the coordinates (the extra coordinate) accounting for thirty orders of magnitude between the size of the observable universe and present upper bounds for the size of extra dimensions.

Another interesting property of our manifold related to the anisotropy of its spatial part is the fact that *any two points* on the brane are separated by a distance of the order of the size of the extra dimension R regardless of their distance measured by means of the induced metric on the brane. Despite the difference in the global topology, the properties of gravity localization turned out to be very similar to the Randall-Sundrum II model, though much more difficult to work out technically. We computed the static (modified) Green's function of a massless scalar field in our background and could show that in the intermediate distance regime $R \ll r \ll R_U$ the 4-dimensional Newton's law is valid for two test particles on the brane, with asymptotic corrections terms identical to the Randall-Sundrum II case up to tiny factors of $\tanh\left(\frac{\omega\pi}{2}\right)$. We could also recover the characteristic 5-dimensional behavior of the Green's function for distances smaller than the extra dimension $r \ll R$. Eventually we saw that in the regime of cosmic distances $r \sim R_U$, somewhat counterintuitive given our highly anisotropic manifold, the Green's function is dominated by the behavior of the corresponding static (modified) Green's function in Einstein's static universe.

In the simple setup considered in this paper the 3-brane is supposed to be motionless. In the light of recent progress in the study of 4-dimensional cosmic evolution induced by the motion of the brane in the bulk, it would be interesting to explore this possibility and see what kind of modifications of our results we would have to envisage. Finally a related, important question which would be interesting to address would be the question of stability of our setup.

Acknowledgments: We wish to thank S. Dubovsky, P. Tinyakov and S. Khlebnikov for useful comments and discussions. E. R. is particularly grateful to E. Teufl for helpful advice in numerous mathematical questions. A. G. wishes to thank LPPC for the kind hospitality during most of this research. This work was supported by the Swiss Science Foundation. A. G. acknowledges "Fondazione A. Della Riccia" for financial support.

A Parallels in the computation of the corrections to Newton's law between the case under consideration and the Randall-Sundrum II case

The purpose of this appendix is to review briefly the calculations of the static two-point function of a scalar field in the Randall-Sundrum II background [47, 49] and to compare each stage with the corresponding stage of calculations in the background considered in this paper. This serves mainly for underlining similarities and differences between the two calculations. Let us begin by writing down the metric of the Randall-Sundrum II model

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \,, \qquad -\pi r_c \leq y \leq \pi r_c \,. \tag{A.1}$$

Here y stands for the extra dimension while k and $\eta_{\mu\nu}$ denote the inverse radius of AdS_5 space and the (4-dimensional)-Minkowski metric with signature $-+++$. Due to the orbifold Z_2-symmetry the allowed range of y is $0 \leq y \leq \pi r_c$.[15] As it is well known, each 4-dimensional graviton mode in this background satisfies the equation of a massless scalar field. Therefore, for the study of the potential between two test masses on the brane we confine ourselves to solving the equation of a massless scalar field with an arbitrary time-independent source:

$$\mathcal{D}u(\vec{x}, y) = j(\vec{x}, y) \qquad \text{with } \mathcal{D} = e^{2ky}\Delta_x - 4k\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2} \,. \tag{A.2}$$

Since the operator $\mathcal{D}$ is formally self-adjoint, the corresponding Green's function will satisfy:

$$\mathcal{D}\mathcal{G}(\vec{x}, \vec{x}'; y, y') = \frac{\delta^3(\vec{x} - \vec{x}')\,\delta(y - y')}{\sqrt{-g}} \,. \tag{A.3}$$

We are now able to write down the usual integral representation of the solution of (A.2):

$$u(\vec{x}, y) = \int \sqrt{-g}\, j(\vec{x}', y')\,\overline{G(\vec{x}, \vec{x}'; y, y')}\, d^3\vec{x}' dy', \tag{A.4}$$

where the y'-integration extends from 0 to πr_c and the $\vec{x}'$ integrations from $-\infty$ to ∞. The absence of boundary terms in (A.4) is of course the result of an appropriate choice of boundary conditions for $u(\vec{x}, y)$ and $G(\vec{x}, \vec{x}'; y, y')$. In the above coordinates of the Randall-Sundrum II case the orbifold boundary conditions together with the Israel condition imposed on the fluctuations of the metric give rise to a Neumann boundary condition at $y = 0$. One can easily convince oneself that in the limit $r_c \to \infty$ the resulting Green's function is independent of the choice of the (homogeneous) boundary condition at $y = \pi r_c$. We therefore follow [6] and use also a Neumann boundary condition at $y = \pi r_c$. Another important point is that the Green's function we are considering is specific to the orbifold boundary condition and so describes the situation of a semi-infinite extra dimension. We decided to carry out the calculations in the semi-infinite case as opposed to [6], where eventually the orbifold boundary conditions are dropped and the case of a fully infinite extra dimension is considered.[16] An immediate consequence of this will be that

[15] We allow for a finite r_c only to impose boundary conditions in a proper way. Eventually we are interested in the limit $r_c \to \infty$.

[16] However as long as matter on the brane is considered the Green's function for the two setups differ only by a factor of 2.

the constant factors of the characteristic short distance singularities of the solutions of Laplace equations also will be modified by a factor of 2. Finally we note that this factor of 2 can be accounted for by an overall redefinition of the 5-dimensional Newton's constant leaving the two theories with equivalent predictions. We conclude the discussion of boundary conditions by noting that at infinity in $\vec{x}'$ we suppose that the $u(\vec{x}', y')$ and $G(\vec{x}, \vec{x}'; y, y')$ vanish sufficiently rapidly.

The solution of eq. (A.3) can be found most conveniently by Fourier expansion in the $\vec{x}$-coordinates and by direct solution of the resulting differential equation for the "transverse" Green's function. After performing the integrations over the angular coordinates in Fourier space we obtain:

$$\mathcal{G}(\vec{x} - \vec{x}', y, y') = \frac{1}{2\pi^2} \int_0^\infty \frac{p \sin\left(p|\vec{x} - \vec{x}'|\right)}{|\vec{x} - \vec{x}'|} g^{(p)}(y, y') dp \,, \tag{A.5}$$

where $g^{(p)}(y, y')$ satisfies

$$\frac{1}{e^{-4ky}} \frac{\partial}{\partial y} \left[e^{-4ky} \frac{\partial}{\partial y} g^{(p)}(y, y') \right] - p^2 e^{2ky} g^{(p)}(y, y') = \frac{\delta(y - y')}{e^{-4ky}} \,, \qquad (0 \le y \le \pi r_c) \tag{A.6}$$

together with Neumann boundary conditions at $y = 0$ and $y = \pi r_c$. The solution of (A.6) is straightforward with the general result

$$\begin{aligned} g^{(p)}(y, y') = {} & \frac{e^{2k(y_> + y_<)}}{k} \times \\ & \times \frac{\left[I_1\left(\frac{p}{k}\right) K_2\left(\frac{p}{k} e^{k y_<}\right) + K_1\left(\frac{p}{k}\right) I_2\left(\frac{p}{k} e^{k y_<}\right) \right] \cdot \left[I_1\left(\frac{p}{k} e^{k\pi r_c}\right) K_2\left(\frac{p}{k} e^{k y_>}\right) + K_1\left(\frac{p}{k} e^{k\pi r_c}\right) I_2\left(\frac{p}{k} e^{k y_>}\right) \right]}{I_1\left(\frac{p}{k}\right) K_1\left(\frac{p}{k} e^{k\pi r_c}\right) - I_1\left(\frac{p}{k} e^{k\pi r_c}\right) K_1\left(\frac{p}{k}\right)} \,, \end{aligned} \tag{A.7}$$

where $y_>$ ($y_<$) denote the greater (smaller) of the two numbers y and y' and $I_1(z), I_2(z), K_1(z), K_2(z)$ are modified Bessel functions. Since we want to study sources on the brane we now set $y' = 0$ and take the well-defined limit $r_c \to \infty$, as can easily be verified from the asymptotic behavior of I_n and K_n for large values of z. We are therefore able to write the static scalar two-point function as

$$\mathcal{G}(\vec{x} - \vec{x}'; y, 0) = -\frac{e^{2ky}}{2\pi^2} \int_0^\infty \frac{\sin\left(p|\vec{x} - \vec{x}'|\right)}{|\vec{x} - \vec{x}'|} \frac{K_2\left(\frac{p}{k} e^{ky}\right)}{K_1\left(\frac{p}{k}\right)} \, dp \,. \tag{A.8}$$

We would like to point out a particularity of the Fourier-integral (A.8). By using the asymptotic expansions for the modified Green's functions $K_1(z)$ and $K_2(z)$ we have [50]:

$$\frac{K_2\left(\frac{p}{k} e^{ky}\right)}{K_1\left(\frac{p}{k}\right)} \sim e^{-\frac{1}{2} ky} e^{-\frac{p}{k}\left(e^{ky} - 1\right)} \left[1 + \mathcal{O}\left(\frac{1}{p}\right) \right] \,, \tag{A.9}$$

showing that for $y = 0$ the integral over p in (A.8) does not exist. The correct value of the Green's function on the brane is therefore obtained by imposing continuity at $y = 0$:

$$\mathcal{G}(\vec{x} - \vec{x}'; 0, 0) \equiv \lim_{y \to 0} \mathcal{G}(\vec{x} - \vec{x}'; y, 0). \tag{A.10}$$

Note that this subtlety is still present in our case of the Fourier sum representation of the modified Green's function (4.62), as discussed in appendix C.

The representation (A.8) is very suitable for obtaining the short distance behavior of the Green's function. By inserting the expansion (A.9) into (A.8) we can evaluate the integral which will give reasonable results for distances smaller than $1/k$:

$$\mathcal{G}(\vec{x}-\vec{x}';y,0) \sim -\frac{e^{\frac{3}{2}ky}}{2\pi^2}\frac{1}{|\vec{x}-\vec{x}'|^2+y^2}\,. \tag{A.11}$$

We see that after replacing the exponential factor by 1 (valid for $y \ll 1/k$) we recover (up to a factor of 2) the correct 5-dimensional behavior of the Green's function in flat 4-dimensional space (4.70).[17]

For large distances the Fourier-representation (A.8) is less suited for obtaining corrections to Newton's law. This is due to the fact that all but the first term in the expansion of K_2/K_1 in powers of p around $p=0$ lead to divergent contributions upon inserting in (A.8). We therefore seek another method which will allow us to obtain the corrections to Newton's law by term-wise integration.

The idea is to promote (A.8) to a contour-integral in the complex p-plane and to shift the contour in such a way that the trigonometric function is transformed into an exponential function. First we introduce dimensionless quantities by rescaling with k according to $X=|\vec{x}-\vec{x}'|k$, $Y=yk$, $z=p|\vec{x}-\vec{x}'|$:

$$\bar{\mathcal{G}}(X,Y,0) = -\frac{k^2e^{2Y}}{2\pi^2X^2}\underbrace{\int_0^\infty \sin z\frac{K_2\left(\frac{z}{X}e^Y\right)}{K_1\left(\frac{z}{X}\right)}dz}_{\equiv I[X,Y]}\,. \tag{A.12}$$

Following [47] (see also [50]) we now use the relation

$$K_2[w] = K_0[w] + \frac{2}{w}K_1[w] \tag{A.13}$$

to separate the zero mode contribution to the Green's function (Newton's law) from the contributions coming from the higher Kaluza-Klein particles (corrections to Newton's law). Using (A.13) in $I[X,Y]$ we find

$$I[X,Y] = \underbrace{\int_0^\infty \sin z\frac{K_0\left(\frac{z}{X}e^Y\right)}{K_1\left(\frac{z}{X}\right)}dz}_{\equiv I_1[X,Y]} + \underbrace{\frac{2X}{e^Y}\int_0^\infty \frac{\sin z}{z}\frac{K_1\left(\frac{z}{X}e^Y\right)}{K_1\left(\frac{z}{X}\right)}dz}_{\equiv I_2[X,Y]}\,. \tag{A.14}$$

The additional factor of $1/z$ in the integrand of $I_2[X,Y]$ allows us to take the limit $Y\to 0$ with the result:

$$\lim_{Y\to 0} I_2[X,Y] = \pi X\,. \tag{A.15}$$

However, we still need $Y>0$ for convergence in $I_1[X,Y]$. The next step is to use $\sin z = \Im\left\{e^{\imath z}\right\}$ and to exchange the operation $\Im$ with the integration over z:[18]

$$I_1[X,Y] = \Im\left\{\int_0^\infty e^{\imath z}\frac{K_0\left(\frac{z}{X}e^Y\right)}{K_1\left(\frac{z}{X}\right)}dz\right\}\,. \tag{A.16}$$

[17] As discussed above the factor of 2 is a direct consequence of the boundary conditions corresponding to a semi-infinite extra dimension.

[18] This is justified since both the real and the imaginary part of the resulting integral converge.

The integrand in (A.16) is a holomorphic function of z in the first quadrant (see [50], p.377 for details). We can therefore apply CAUCHY's theorem to the contour depicted in Fig. 3. As we will

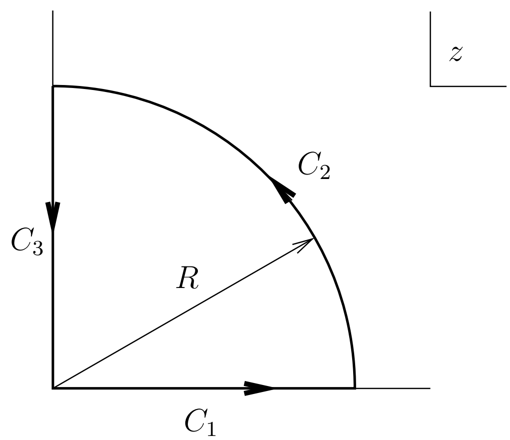

Figure 3: Contour used in CAUCHY's theorem to evaluate the integral (A.16).

show shortly, the contribution from the arc $C_2 = \left\{z \,|\, z = Re^{\imath\varphi},\ 0 \le \varphi \le \frac{\pi}{2}\right\}$ vanishes in the limit $R \to \infty$ (the limit we are interested in). Symbolically we therefore have

$$\lim_{R\to\infty} \left\{ \int_{C_1} \ldots + \int_{C_3} \ldots \right\} = 0\,, \tag{A.17}$$

where ... replace the integrand in (A.16) and the sense of integration is as indicated in the figure. This means that we can replace the integration over the positive real axis in (A.16) by an integration over the positive imaginary axis without changing the value of $I_1[X, Y]$. Substituting now z in favor of n according to $n = -\imath z$ we obtain:

$$\begin{aligned} I_1[X, Y] &= \Im \left\{ \int_0^\infty e^{-n} \frac{K_0\left(\frac{\imath n}{X} e^Y\right)}{K_1\left(\frac{\imath n}{X}\right)} \imath dn \right\} = \Im \left\{ \int_0^\infty e^{-n} \frac{-H_0^{(2)}\left(\frac{n}{X} e^Y\right)}{H_1^{(2)}\left(\frac{n}{X}\right)} dn \right\} \\ &= \int_0^\infty e^{-n} \frac{Y_0\left(\frac{n}{X} e^Y\right) J_1\left(\frac{n}{X}\right) - J_0\left(\frac{n}{X} e^Y\right) Y_1\left(\frac{n}{X}\right)}{\left[J_1\left(\frac{n}{X}\right)\right]^2 + \left[Y_1\left(\frac{n}{X}\right)\right]^2} dn\,. \end{aligned} \tag{A.18}$$

In the first line in (A.18) we replaced modified Bessel functions with imaginary arguments by Hankel functions of real argument. In the second line we explicitly took the imaginary part after replacing the identities relating the Hankel functions and the Bessel functions J and Y of the first kind. The benefit from the rotation of the contour of integration is obvious at this stage. First, the last integral in (A.18) converges also for $Y = 0$ due to the presence of the exponential function in the integrand. In this case we have

$$I_1[X, 0] = \frac{2X}{\pi} \int_0^\infty \frac{e^{-n}}{n} \frac{1}{\left[J_1\left(\frac{n}{X}\right)\right]^2 + \left[Y_1\left(\frac{n}{X}\right)\right]^2} dn\,. \tag{A.19}$$

Note that we used the Wronskian relation for Bessel functions to simplify the numerator, the general reference being once more [50]. Second, it is straightforward to obtain the large X asymptotic

($1 \ll X$) of the integral (A.19) by a simple power series expansion in n around $n = 0$ of the integrand (apart from the exponential factor), followed by term-wise integration. For the results see [46], where the above integral (A.19) has been found in the wave-function approach:

$$I_1[X,0] \sim \frac{\pi}{2X} + \frac{\pi}{X^3}\left(4 - 3\ln 2 - 3\ln X\right) + \mathcal{O}\left[\frac{\ln X}{X^5}\right]. \tag{A.20}$$

The result for $\bar{\mathcal{G}}(X,0,0)$ is therefore:

$$\bar{\mathcal{G}}(X,0,0) = -\frac{k^2}{2\pi X}\left[1 + \frac{1}{2X^2} + \frac{1}{X^4}\left(4 - 3\ln 2 - 3\ln X\right) + \mathcal{O}\left(\frac{\ln X}{X^6}\right)\right]. \tag{A.21}$$

However, we still have to verify that the contribution from the arc C_2 in Fig. 3 vanishes in the limit $R \to \infty$. Parametrizing z by $z = Re^{\imath\varphi}$, $0 \le \varphi \le \pi/2$ we find:

$$\begin{aligned} \left|\int_{C_2} e^{\imath z}\frac{K_0\left(\frac{z}{X}e^Y\right)}{K_1\left(\frac{z}{X}\right)}dz\right| &= \left|\int_0^{\frac{\pi}{2}} e^{\imath Re^{\imath\varphi}}\frac{K_0\left(\frac{Re^{\imath\varphi}}{X}e^Y\right)}{K_1\left(\frac{Re^{\imath\varphi}}{X}\right)}\left(\imath Re^{\imath\varphi}\right)d\varphi\right| \\ &\le R\int_0^{\frac{\pi}{2}}\left|e^{\imath R(\cos\varphi + \imath\sin\varphi)}\right| \cdot \left|\frac{K_0\left(\frac{Re^{\imath\varphi}}{X}e^Y\right)}{K_1\left(\frac{Re^{\imath\varphi}}{X}\right)}\right| d\varphi\,. \end{aligned} \tag{A.22}$$

Making now use of the asymptotic properties of the modified Bessel functions to estimate the second modulus in the last integrand in (A.22) we can write:

$$\left|\frac{K_0\left(\frac{Re^{\imath\varphi}}{X}e^Y\right)}{K_1\left(\frac{Re^{\imath\varphi}}{X}\right)}\right| \le C\left|e^{-\frac{Y}{2}}e^{-Re^{\imath\varphi}\frac{e^Y-1}{X}}\right|, \tag{A.23}$$

so that

$$\left|\int_{C_2} e^{\imath z}\frac{K_0\left(\frac{z}{X}e^Y\right)}{K_1\left(\frac{z}{X}\right)}dz\right| < CRe^{-\frac{Y}{2}}\int_0^{\frac{\pi}{2}} e^{-R\left[\sin\varphi + \left(\frac{e^Y-1}{X}\right)\cos\varphi\right]}d\varphi\,, \tag{A.24}$$

with C being a constant of the order of unity, independent of R and φ. To finish the estimate we observe that

$$\sin\varphi + \left(\frac{e^Y-1}{X}\right)\cos\varphi \ge \epsilon \equiv \min\left[1, \frac{e^Y-1}{X}\right] > 0 \quad , \quad \forall\,\varphi \in [0, \frac{\pi}{2}]\,, \qquad (Y > 0) \tag{A.25}$$

and obtain:

$$\left|\int_{C_2} e^{\imath z}\frac{K_0\left(\frac{z}{X}e^Y\right)}{K_1\left(\frac{z}{X}\right)}dz\right| < \frac{CR\pi}{2}e^{-\frac{Y}{2}}e^{-R\epsilon} \to 0 \quad \text{for } R \to \infty. \tag{A.26}$$

This establishes the result that in the limit $R \to \infty$ the arc C_2 does not contribute to the integral in (A.16) and completes our discussion of the two-point function in the Randall Sundrum-II setup.

We are now going to summarize briefly the complications arising when the general scheme of computations outlined above for the Randall-Sundrum II case are applied to the brane setup considered in this paper. First and foremost, the main difference is due to the fact that the induced metric on our brane is not the flat Minkowski metric but the metric of the Einstein static universe. This implies the use of a modified Green's function. In the eigenfunction expansion, the discrete scalar-harmonics on S^3 will replace the continuous plane-wave eigenfunctions. We therefore expect the formal solution for the two-point function (the analog of (A.5)) to take the form of a Fourier sum.

The solution to the *transverse* Green's function (the analog of eq. (A.6)) as presented in appendix B turns out to be again straightforward and completes the formal solution.

When trying to distinguish between the zero mode contribution and the contribution coming from higher modes (in the language of the Kaluza-Klein approach) a relation similar to (A.13) proves useful. While the evaluation of the zero mode contribution poses no problems, the corrections coming from higher modes are much more involved this time. We face the following major technical difficulties: there is no analog of CAUCHY's theorem in the discrete case of the Fourier sum. One solution to this problem is to use a variant of EULER-MACLAURIN's sum rule in order to replace the Fourier sum by an analog Fourier integral and a remainder term also in the form of an integral. Now the procedure again is similar to the one outlined for the Randall-Sundrum case. Another technical problem, however, in connection with large distances corrections has its origin in the simple fact that by large distances in our setup we mean distances large with respect to the extra dimension R but also small with respect to the size of the observable universe R_U. The detailed computations of the corrections in this distances regime can be found in appendix C.

What concerns the computation of the two-point function at distances smaller than the extra dimension, there is no conceptual difference to the Randall-Sundrum case. The Fourier sum can be calculated analytically after inserting the corresponding large momentum asymptotic. We give the details in appendix D.

B Solutions to the differential equations (4.55) and (4.60)

The aim of this appendix is to obtain the solutions to the eqs. (4.55) and (4.60) and hence to complete the computation of the modified Green's function. We will concentrate mainly on (4.60) for two reasons: first and foremost because in the formal expansion (4.62) $g^{(1)}(\theta,\theta')$ multiplies a constant function in s and since we are mainly interested in the behavior of the Green's function on the brane, this constant is of no relevance for our considerations. Secondly because the construction of the Green's functions $g^{(1)}(\theta,\theta')$ and $g^{(\lambda)}(\theta,\theta')$ follows the standard procedure, so it is not necessary to go into details twice. We will merely state the result for $g^{(1)}(\theta,\theta')$. The equation we would like to solve is

$$\frac{1-\lambda^2}{R_U^2\gamma(\theta)^2}g^{(\lambda)}(\theta,\theta')+\frac{1}{R^2}\frac{1}{\sigma(\theta)\gamma(\theta)^3}\frac{\partial}{\partial\theta}\left[\sigma(\theta)\gamma(\theta)^3\frac{\partial g^{(\lambda)}(\theta,\theta')}{\partial\theta}\right]=\frac{\delta(\theta-\theta')}{\sigma(\theta)\gamma(\theta)^3}\,, \tag{B.1}$$

where $\lambda=2,3,4\ldots$ and the independent variables θ and θ' are restricted to the intervals $-\pi/2\leq\theta\leq\pi/2$, and $-\pi/2\leq\theta'\leq\pi/2$. The homogeneous equation associated with (B.1) can be reduced to the hypergeometric equation. Its general formal solution in the interval $0\leq\theta\leq\pi/2$ is given by

$$\varphi^{(+)}(\lambda,\theta)=a_1^{(+)}\varphi_1(\lambda,\theta)+a_2^{(+)}\varphi_2(\lambda,\theta)\,,\qquad 0\leq\theta\leq\frac{\pi}{2} \tag{B.2}$$

where

$$\begin{aligned}\varphi_1(\lambda,\theta)&=\frac{\tanh^{\lambda-1}\left[\omega\left(\frac{\pi}{2}-\theta\right)\right]}{\cosh^4\left[\omega\left(\frac{\pi}{2}-\theta\right)\right]}\ {}_2F_1\left[\frac{\lambda+3}{2},\frac{\lambda+3}{2};\lambda+1;\tanh^2\left[\omega\left(\frac{\pi}{2}-\theta\right)\right]\right]\,,\\ \varphi_2(\lambda,\theta)&=\frac{\tanh^{\lambda-1}\left[\omega\left(\frac{\pi}{2}-\theta\right)\right]}{\cosh^4\left[\omega\left(\frac{\pi}{2}-\theta\right)\right]}\ {}_2F_1\left[\frac{\lambda+3}{2},\frac{\lambda+3}{2};3;\cosh^{-2}\left[\omega\left(\frac{\pi}{2}-\theta\right)\right]\right]\,.\end{aligned} \tag{B.3}$$

Due to the symmetry of the background under $\theta\to-\theta$ we can immediately write down the general solution of the homogeneous equation obtained from (B.1) in the interval $-\pi/2\leq\theta\leq0$:

$$\varphi^{(-)}(\lambda,\theta)=a_1^{(-)}\varphi_1(\lambda,-\theta)+a_2^{(-)}\varphi_2(\lambda,-\theta)\,,\qquad -\frac{\pi}{2}\leq\theta\leq0\,. \tag{B.4}$$

We now have to decide whether we want to restrict our Green's function to describe perturbations which also possess the symmetry $\theta\to-\theta$ of the background (as in the Randall-Sundrum case) or not. In the first case one can limit the solution of (B.1) to $\theta>0$ and continue the result symmetrically into $\theta<0$. Note that imposing the symmetry $\theta\to-\theta$ on the perturbations goes hand in hand with imposing $\theta\to-\theta$ for the source which means that one should add a corresponding delta-function $\delta(\theta+\theta')$ to the right hand side of (B.1). The Israel-condition (4.14) on the brane with $\theta=0$ then gives $\frac{\partial g^{(\lambda)}}{\partial\theta}=0$, as in the Randall-Sundrum case.

Since in our setup we do not have any convincing argument for imposing the $\theta\to-\theta$ symmetry also on the metric fluctuations (the scalar field), the second case, which allows a breaking of this symmetry will be the case of our choice. The Israel-condition (4.14) now merely indicates that the perturbation (the scalar field) should have a continuous first derivative at the brane.

In order to obtain a uniquely defined solution we have to impose two boundary conditions on $g^{(\lambda)}(\theta,\theta')$ one at $\theta=\pi/2$ the other at $\theta=-\pi/2$. It turns out that the requirement of square integrability of our solutions (with the correct weight-function) forces us to discard one of the two fundamental solutions at the boundaries $\theta=\pm\pi/2$, namely $\varphi_2(\lambda,\theta)$.

Before going into details we introduce the following abbreviations:

$$p(\theta) = \frac{\sigma(\theta)\gamma(\theta)^3}{R^2}\,, \qquad s(\theta) = \frac{\sigma(\theta)\gamma(\theta)}{R_U^2}\,. \tag{B.5}$$

Eq. (B.1) then becomes

$$\left[p(\theta) g^{(\lambda)\prime}(\theta,\theta')\right]' - (\lambda^2-1)\, s(\theta) g^{(\lambda)}(\theta,\theta') = \delta(\theta-\theta')\,, \tag{B.6}$$

where $'$ denotes derivatives with respect to θ here and in the following. We start by taking $\theta' > 0$. The case $\theta' < 0$ is fully analogous to the one considered up to minus signs. Our ansatz for $g^{(\lambda)}(\theta,\theta')$ is

$$g^{(\lambda)}(\theta,\theta') = \begin{cases} A(\theta')\,\varphi_1^{(\lambda)}(-\theta) & -\frac{\pi}{2} \le \theta \le 0\,, \\ B(\theta')\,\varphi_1^{(\lambda)}(\theta) + C(\theta')\,\varphi_2^{(\lambda)}(\theta) & 0 \le \theta \le \theta'\,, \\ D(\theta')\,\varphi_1^{(\lambda)}(\theta) & \theta' \le \theta \le \frac{\pi}{2}\,. \end{cases} \tag{B.7}$$

We impose the following boundary and matching conditions on $g^{(\lambda)}(\theta,\theta')$ which will uniquely determine the coefficients A, B, C, D:

1. continuity of $g^{(\lambda)}(\theta,\theta')$ at $\theta = 0$.
2. continuity of $g^{(\lambda)\prime}(\theta,\theta')$ at $\theta = 0$.
3. continuity of $g^{(\lambda)}(\theta,\theta')$ at $\theta = \theta'$.
4. jump condition of $g^{(\lambda)\prime}(\theta,\theta')$ at $\theta = \theta'$.

Using the ansatz (B.7) we obtain from the above conditions:

$$\begin{array}{rcrcl} [A(\theta') - B(\theta')]\,\varphi_1^{(\lambda)}(0) & - & C(\theta')\varphi_2^{(\lambda)}(0) & = & 0\,, \\ [A(\theta') + B(\theta')]\,\varphi_1^{(\lambda)\prime}(0) & + & C(\theta')\varphi_2^{(\lambda)\prime}(0) & = & 0\,, \\ [B(\theta') - D(\theta')]\,\varphi_1^{(\lambda)}(\theta') & + & C(\theta')\varphi_2^{(\lambda)}(\theta') & = & 0\,, \\ [B(\theta') - D(\theta')]\,\varphi_1^{(\lambda)\prime}(\theta') & + & C(\theta')\varphi_2^{(\lambda)\prime}(\theta') & = & -\frac{1}{p(\theta')}\,, \end{array} \tag{B.8}$$

the solution of which is easily found to be:

$$\begin{aligned} A(\theta') &= \frac{R^2}{2}\frac{\varphi_1^{(\lambda)}(\theta')}{\varphi_1^{(\lambda)}(0)\varphi_1^{(\lambda)\prime}(0)}\,, \quad B(\theta') = \frac{R^2}{2}\frac{\varphi_1^{(\lambda)}(\theta')}{\mathcal{W}\left[\varphi_1^{(\lambda)},\varphi_2^{(\lambda)},0\right]}\frac{\varphi_1^{(\lambda)}(0)\varphi_2^{(\lambda)\prime}(0) + \varphi_2^{(\lambda)}(0)\varphi_1^{(\lambda)\prime}(0)}{\varphi_1^{(\lambda)}(0)\varphi_1^{(\lambda)\prime}(0)}\,, \\ C(\theta') &= -R^2\frac{\varphi_1^{(\lambda)}(\theta')}{\mathcal{W}\left[\varphi_1^{(\lambda)},\varphi_2^{(\lambda)},0\right]}\,, \qquad D(\theta') = B(\theta') - R^2\frac{\varphi_2^{(\lambda)}(\theta')}{\mathcal{W}\left[\varphi_1^{(\lambda)},\varphi_2^{(\lambda)},0\right]}\,, \end{aligned} \tag{B.9}$$

where $\mathcal{W}\left[\varphi_1^{(\lambda)},\varphi_2^{(\lambda)},0\right]$ denotes the Wronskian of $\varphi_1^{(\lambda)}$ and $\varphi_2^{(\lambda)}$ at $\theta = 0$. In obtaining (B.9) we used the relation

$$\mathcal{W}\left[\varphi_1^{(\lambda)},\varphi_2^{(\lambda)},\theta\right] = \frac{\mathcal{W}\left[\varphi_1^{(\lambda)},\varphi_2^{(\lambda)},0\right]}{R^2\,p(\theta)}\,. \tag{B.10}$$

As already mentioned, the case $\theta' < 0$ can be treated in complete analogy to the case $\theta' > 0$. Combining the two results gives the final expression for the solution $g^{(\lambda)}(\theta,\theta')$ of (B.1) (for $\lambda = 2,3,\ldots$):

$$g^{(\lambda)}(\theta,\theta') = \begin{cases} A(|\theta'|)\,\varphi_1^{(\lambda)}(|\theta|) & -\frac{\pi}{2} \leq \theta\,\mathrm{sign}(\theta') \leq 0\,, \\ B(|\theta'|)\,\varphi_1^{(\lambda)}(|\theta|) + C\,(|\theta'|)\varphi_2^{(\lambda)}(|\theta|) & 0 \leq \theta\,\mathrm{sign}(\theta') \leq |\theta'|\,, \\ D(|\theta'|)\,\varphi_1^{(\lambda)}(|\theta|) & |\theta'| \leq \theta\,\mathrm{sign}(\theta') \leq \frac{\pi}{2}\,. \end{cases} \tag{B.11}$$

It is easily verified that the Green's function (B.11) has the property

$$g^{(\lambda)}(\theta,\theta') = g^{(\lambda)}(\theta',\theta) \tag{B.12}$$

as expected for a self-adjoint boundary value problem. We will not need the result (B.11) in its full generality. We focus on the case where sources are located at the brane ($\theta' = 0$). In this case (B.11) reduces to:

$$g^{(\lambda)}(\theta,0) = \frac{R^2}{2}\frac{\varphi_1^{(\lambda)}(|\theta|)}{\varphi_1^{(\lambda)}{}'(0)}\,, \qquad (\lambda = 2,3,\ldots)\,. \tag{B.13}$$

For the sake of completeness, we also give the solution $g^{(1)}(\theta,\theta')$ to (4.55)

$$\frac{1}{R^2}\frac{1}{\sigma(\theta)\gamma(\theta)^3}\frac{\partial}{\partial\theta}\left[\sigma(\theta)\gamma(\theta)^3\frac{\partial g^{(1)}(\theta,\theta')}{\partial\theta}\right] = \frac{1}{\sigma(\theta)\gamma(\theta)^3}\delta(\theta-\theta') - \tilde{\chi}_1(\theta)\bar{\tilde{\chi}}_1(\theta')\,. \tag{B.14}$$

The general procedure of finding the solution is fully analogous to the case of $g^{(\lambda)}(\theta,\theta')$ (for $\lambda = 2,3,\ldots$), the only difference being that in regions where $\theta \neq \theta'$ (B.14) reduces to an inhomogeneous differential equation. In the interval $0 \leq \theta \leq \pi/2$ the general solution is given by

$$\psi(\theta) = \psi_0(\theta) + A\psi_1(\theta) + B\psi_2(\theta), \tag{B.15}$$

with

$$\begin{aligned} \psi_0(\theta) &= -\frac{R^2}{2\omega\tanh\left(\frac{\omega\pi}{2}\right)}\ln\left[\cosh\left[\omega\left(\frac{\pi}{2}-\theta\right)\right]\right]\,, \\ \psi_1(\theta) &= 1\,, \qquad \psi_2(\theta) = 2\ln\left[\tanh\left[\omega\left(\frac{\pi}{2}-\theta\right)\right]\right] + \coth^2\left[\omega\left(\frac{\pi}{2}-\theta\right)\right]\,. \end{aligned} \tag{B.16}$$

With these definitions, it is easy to verify that $g^{(1)}(\theta,\theta')$ is given by:

$$g^{(1)}(\theta,\theta') = \text{const.} + \psi_0(|\theta|) + \psi_0(|\theta'|) + C\begin{cases} \psi_2(0) & -\frac{\pi}{2} \leq \theta\,\mathrm{sign}(\theta') \leq 0 \\ \psi_2(|\theta|) & 0 \leq \theta\,\mathrm{sign}(\theta') \leq |\theta'| \\ \psi_2(|\theta'|) & |\theta'| \leq \theta\,\mathrm{sign}(\theta') \leq \frac{\pi}{2} \end{cases}\,, \tag{B.17}$$

with

$$C = -\frac{R^2}{\mathcal{W}\left[\psi_1,\psi_2,0\right]}\,. \tag{B.18}$$

By $\mathcal{W}\left[\psi_1,\psi_2,0\right]$ we again mean the Wronskian of ψ_1 and ψ_2 evaluated at $\theta = 0$.

C Detailed computation of the corrections to Newton's law

In this appendix we give a detailed computation of the sum

$$S[s,\theta,\omega]=\sum_{\lambda=2}^{\infty}\lambda\sin(\lambda s)\,\frac{\varphi_1^{(\lambda)}(\theta)}{\varphi'^{(\lambda)}_1(0)}\,,\qquad 0<\theta\leq\frac{\pi}{2} \tag{C.1}$$

with

$$\frac{\varphi_1^{(\lambda)}(\theta)}{\varphi'^{(\lambda)}_1(0)} \;=\; -\frac{z(\theta)^{\frac{\lambda-1}{2}}\left[1-z(\theta)\right]^2\ {}_2F_1\left[\frac{\lambda+3}{2},\frac{\lambda+3}{2};\lambda+1;z(\theta)\right]}{\omega(\lambda-1)z(0)^{\frac{\lambda-2}{2}}\left[1-z(0)\right]^2\ {}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda+3}{2};\lambda+1;z(0)\right]}\,, \tag{C.2}$$

where we explicitly excluded $\theta=0$ since the Fourier sum (C.1) does not converge for this value of θ.[19] We also used the definition $z(\theta)=\tanh^2\left[\omega\left(\frac{\pi}{2}-\theta\right)\right]$.

Note that the sum $S[s,\theta,\omega]$ contains the contribution to the two-point function coming from zero mode and higher Kaluza-Klein modes. Inspired by the Randall-Sundrum case we try to separate the two contributions by functional relations between contiguous Gauss Hypergeometric functions. Using (15.2.15) of reference [50] for $a=b-1=(\lambda+1)/2$, $c=\lambda+1$ we obtain

$$\begin{aligned}\left[1-z(\theta)\right]\ {}_2F_1\left[\frac{\lambda+3}{2},\frac{\lambda+3}{2};\lambda+1;z(\theta)\right] &= \frac{2}{\lambda+1}\ {}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda+3}{2};\lambda+1;z(\theta)\right]\\ &+ \frac{\lambda-1}{\lambda+1}\ {}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda+1}{2};\lambda+1;z(\theta)\right]\,.\end{aligned} \tag{C.3}$$

In this way we are able to rewrite $S[s,\theta,\omega]$ in the form:

$$S[s,\theta,\omega]=S_1[s,\theta,\omega]+S_2[s,\theta,\omega]\,, \tag{C.4}$$

where

$$S_1[s,\theta,\omega]=-\frac{2}{\omega}\frac{z(0)}{z(\theta)^{\frac{1}{2}}}\frac{1-z(\theta)}{\left[1-z(0)\right]^2}\sum_{\lambda=2}^{\infty}\frac{\lambda\sin(\lambda s)}{\lambda^2-1}\left[\frac{z(\theta)}{z(0)}\right]^{\frac{\lambda}{2}}\ \frac{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda+3}{2};\lambda+1;z(\theta)\right]}{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda+3}{2};\lambda+1;z(0)\right]}\,, \tag{C.5}$$

$$S_2[s,\theta,\omega]=-\frac{1}{\omega}\frac{z(0)}{z(\theta)^{\frac{1}{2}}}\frac{1-z(\theta)}{\left[1-z(0)\right]^2}\sum_{\lambda=2}^{\infty}\frac{\lambda\sin(\lambda s)}{\lambda+1}\left[\frac{z(\theta)}{z(0)}\right]^{\frac{\lambda}{2}}\ \frac{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda+1}{2};\lambda+1;z(\theta)\right]}{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda+3}{2};\lambda+1;z(0)\right]}\,. \tag{C.6}$$

From the asymptotic formula (D.1) we infer that now the sum $S_1[s,\theta,\omega]$ is convergent even for $\theta=0$ due to the additional power of λ in the denominator. We therefore obtain

$$S_1[s,0,\omega]=-\frac{2}{\omega}\frac{z(0)^{\frac{1}{2}}}{1-z(0)}\sum_{\lambda=2}^{\infty}\frac{\lambda\sin(\lambda s)}{\lambda^2-1}=-\frac{2}{\omega}\frac{z(0)^{\frac{1}{2}}}{1-z(0)}\left(\frac{\pi-s}{2}\cos s-\frac{1}{4}\sin s\right)\,. \tag{C.7}$$

Since we recognized in the last sum the two point function of Einstein's static universe, we attribute the contribution coming from $S_1[s,\theta,\omega]$ to the zero mode in the Kaluza-Klein spectrum. We also

[19] Note that this mathematical delicacy about the divergence of the Green's function on the brane also applies to the Randall-Sundrum case of the Fourier integral representation of the two-point function. The obvious remedy is to keep θ strictly greater than zero during the whole calculation and eventually take the limit $\theta\to 0$.

should mention that in $S_2[s,\theta,\omega]$ we still need $\theta > 0$ for convergence since only then the factor $[z(\theta)/z(0)]^{\frac{\lambda-1}{2}}$ provides an exponential cutoff in λ for the sum.

The main challenge in the evaluation of the two-point function is therefore to tame the sum $S_2[s,\theta,\omega]$. Our strategy is the following: first we extend it from $\lambda = 0$ to ∞ by adding and subtracting the $\lambda = 0$ and $\lambda = 1$ terms. Next, we replace the sum over λ by the sum of two integrals employing a variant of the Euler-Maclaurin sum rule called the Abel-Plana formula (see. e.g. [51], p. 289-290). Finally, we shall see that the resulting (exact) integral representation will allow us to extract the desired asymptotic of the two-point function on the brane in the distance regime of interest ($R \ll r \ll R_U$).

As announced, we start by extending the range of the sum from 0 to ∞. Since the addend with $\lambda = 0$ vanishes, we only need to subtract the $\lambda = 1$ term with the result:

$$S_2[s,\theta,\omega] = -\frac{1}{2\omega}\frac{z(0)^{\frac{1}{2}}}{z(\theta)}\frac{1-z(\theta)}{1-z(0)}\ln\left[1-z(\theta)\right]\sin s - \frac{1}{\omega}\frac{z(0)}{z(\theta)^{\frac{1}{2}}}\frac{1-z(\theta)}{1-z(0)}R[s,\theta,\omega]\,, \tag{C.8}$$

where $R[s,\theta,\omega]$ is defined by

$$R[s,\theta,\omega] = \sum_{\lambda=0}^{\infty}\frac{\lambda\sin(\lambda s)}{\lambda+1}\left[\frac{z(\theta)}{z(0)}\right]^{\frac{\lambda}{2}}\frac{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda+1}{2};\lambda+1;z(\theta)\right]}{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda-1}{2};\lambda+1;z(0)\right]}\,. \tag{C.9}$$

We used the identity

$${}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda+3}{2};\lambda+1;z(0)\right] = \left[1-z(0)\right]^{-1}\ {}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda-1}{2};\lambda+1;z(0)\right] \tag{C.10}$$

in the last step (see 15.3.3 of [50]). The next step consists of extending the sin function in (C.9) to an exponential and taking the imaginary part out of the sum (as in the Randall-Sundrum case). Now we make use of the Abel-Plana formula, first considering the partial sums:

$$\begin{aligned} R^{(n)}[s,\theta,\omega] \;=\;& \Im\Bigg[\frac{1}{2}f(0,s,\theta,\omega)+\frac{1}{2}f(n,s,\theta,\omega)+\int_0^n f(\lambda,s,\theta,\omega)\,d\lambda \qquad\qquad \text{(C.11)} \\ & +\imath\int_0^{\infty}\frac{f(\imath y,s,\theta,\omega)-f(n+\imath y,s,\theta,\omega)-f(-\imath y,s,\theta,\omega)+f(n-\imath y,s,\theta,\omega)}{e^{2\pi y}-1}dy\Bigg]\,, \end{aligned}$$

where for the sake of clarity we introduced

$$f(\lambda,s,\theta,\omega) \equiv \frac{\lambda e^{\imath\lambda s}}{\lambda+1}\left[\frac{z(\theta)}{z(0)}\right]^{\frac{\lambda}{2}}\frac{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda+1}{2};\lambda+1;z(\theta)\right]}{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda-1}{2};\lambda+1;z(0)\right]}. \tag{C.12}$$

In the limit $n \to \infty$ we find:

$$R[s,\theta,\omega] \;=\; \Im\Big[\underbrace{\int_0^{\infty}f(\lambda,s,\theta,\omega)d\lambda}_{\equiv R_1[s,\theta,\omega]}+\underbrace{\imath\int_0^{\infty}\frac{f(\imath y,s,\theta,\omega)-f(-\imath y,s,\theta,\omega)}{e^{2\pi y}-1}dy}_{\equiv R_2[s,\theta,\omega]}\Big]\,. \tag{C.13}$$

C.1 Leading order asymptotic of $\Im\{R_1[s,\theta,\omega]\}$

We focus first on $R_1[s,\theta,\omega]$ and observe that the function $f(\lambda,s,\theta,\omega)$ is holomorphic (in λ) in the first quadrant.[20] Therefore, in perfect analogy with the Randall-Sundrum case treated in appendix A, we can use CAUCHY's theorem to replace the integration over the positive real axis by an integration over the positive imaginary axis.[21] After substituting λ with $\imath y$ we obtain :

$$R_1[s,\theta,\omega] = -\int_0^\infty \frac{y(1-\imath y)}{1+y^2} e^{-ys} \left[\frac{z(\theta)}{z(0)}\right]^{\frac{\imath y}{2}} \frac{{}_2F_1\left[\frac{1+\imath y}{2},\frac{1+\imath y}{2};1+\imath y;z(\theta)\right]}{{}_2F_1\left[\frac{1+\imath y}{2},\frac{-1+\imath y}{2};1+\imath y;z(0)\right]} dy\,. \tag{C.14}$$

The upper deformation of the path of integration is advantageous for at least two reasons. Firstly, through the appearance of the exponential in the integrand, convergence on the brane does no longer rely on $\theta>0$ and we can set θ equal to zero in (C.14) to obtain:

$$R_1[s,0,\omega] = -\int_0^\infty \frac{y(1-\imath y)}{1+y^2} e^{-ys} \frac{{}_2F_1\left[\frac{1+\imath y}{2},\frac{1+\imath y}{2};1+\imath y;z(0)\right]}{{}_2F_1\left[\frac{1+\imath y}{2},\frac{-1+\imath y}{2};1+\imath y;z(0)\right]} dy\,. \tag{C.15}$$

Secondly, the form (C.14) is perfectly suited for obtaining asymptotic expansions for large distances. Note that since $s\in[0,\pi]$, we carefully avoided saying "for large s" since this would not correspond to the case of our interest. We try to compute the two-point function in a distance regime $R \ll r \ll R_U$, distances clearly far beyond the size of the fifth dimension but far below the size of the observable universe. Rewritten in the geodesic distance coordinate s this becomes $\omega/\sinh\left(\frac{\omega\pi}{2}\right) \ll s \ll 1$. It is this last relation which complicates the computation of the asymptotic considerably. We are forced to look at an asymptotic evaluation of (C.15) (or (C.14)) for intermediate s values such that we cannot make direct use of Laplace's method. The correct way of extracting the above described asymptotic is to expand the ratio of hypergeometric functions in (C.15) in a power series of the small parameter $1-z(0)$. Using rel. (15.3.10) and (15.3.11) (with $m=1$) of [50] we find after some algebra

$$\begin{aligned}
\frac{{}_2F_1\left[\frac{1+\imath y}{2},\frac{1+\imath y}{2};1+\imath y;z(0)\right]}{{}_2F_1\left[\frac{1+\imath y}{2},\frac{-1+\imath y}{2};1+\imath y;z(0)\right]} &= -\frac{1}{2}(1+\imath y)\left[2\gamma+2\psi(\frac{1+\imath y}{2})+\ln\left[1-z(0)\right]\right] \\
+ \quad & \left\{\left(\frac{1+\imath y}{2}\right)^3\left[2\psi(2)-2\psi\left(\frac{3+\imath y}{2}\right)-\ln\left[1-z(0)\right]\right]\right. \\
& -\left(\frac{1+\imath y}{2}\right)^2\left(\frac{-1+\imath y}{2}\right)\left[2\psi(1)-2\psi\left(\frac{1+\imath y}{2}\right)-\ln\left[1-z(0)\right]\right]\times \\
& \quad\times\left.\left[\ln\left[1-z(0)\right]-\psi(1)-\psi(2)+\psi\left(\frac{3+\imath y}{2}\right)+\psi\left(\frac{1+\imath y}{2}\right)\right]\right\}\left[1-z(0)\right] \\
+ \quad & \mathcal{O}\left\{\ln\left[1-z(0)\right]\left[1-z(0)\right]^2\right\}\,.
\end{aligned} \tag{C.16}$$

[20] Everything but the ratio of the hypergeometric functions is clearly holomorphic. The dependence of Gauss's hypergeometric function on the parameters is also holomorphic so the only danger comes from the zeros of ${}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda-1}{2};\lambda+1;z(0)\right]$. These, however, turn out to be outside of the first quadrant.

[21] To be mathematically correct, one has to apply CAUCHY's theorem to the quarter of the disk of radius R bounded by the positive real axis, the positive imaginary axis and the arc joining the points R and $\imath R$, see Fig. 3 in appendix A. The above statement will then be correct if in the limit $R\to\infty$ the contribution from the arc tends to zero. We will verify this in section C.4 of this appendix.

In the above relation γ denotes EULER-MASCHERONI's constant and $\psi(z) \equiv \Gamma'[z]/\Gamma[z]$ the so-called Digamma-function. Note that we developed up to linear order in $1 - z(0)$ since we also want to compute the next to leading order term in the asymptotic later in this appendix.

We are only interested in the imaginary part of $R_1[s, 0, \omega]$ and it turns out that the first term in the expansion (C.16) after insertion in (C.15) can be integrated analytically:

$$\Im\left\{R_1^{(0)}[s, 0, \omega]\right\} = \Im\left\{\int_0^\infty y e^{-ys}\psi\left(\frac{1+\imath y}{2}\right) dy\right\} = \int_0^\infty y e^{-ys}\Im\left[\psi\left(\frac{1+\imath y}{2}\right)\right] dy\,. \tag{C.17}$$

The last step is justified since also the real part of the first integral in (C.17) converges, as follows immediately from the asymptotic $\psi(z) \sim \ln z - \frac{1}{2z} + \mathcal{O}(\frac{1}{z^2})$. Since

$$\Im\left[\psi(\frac{1+\imath y}{2})\right] = \frac{\pi}{2}\tanh\left(\frac{\pi y}{2}\right)\,, \tag{C.18}$$

([50], p.259, 6.3.12) we find

$$\begin{aligned}
\Im\left\{R_1^{(0)}[s, 0, \omega]\right\} &= \frac{\pi}{2}\int_0^\infty y e^{-ys}\tanh\left(\frac{\pi y}{2}\right) dy = -\frac{\pi}{2}\frac{d}{ds}\left[\int_0^\infty e^{-ys}\tanh\left(\frac{\pi y}{2}\right) dy\right] \\
&= -\frac{\pi}{2}\frac{d}{ds}\left[\int_0^\infty e^{-ys}\left(\frac{2}{1+e^{-\pi y}} - 1\right) dy\right] = -\frac{\pi}{2}\frac{d}{ds}\left[-\frac{1}{s} + 2\int_0^\infty \frac{e^{-ys}}{1+e^{-\pi y}} dy\right] \\
&= -\frac{\pi}{2s^2} - \frac{d}{ds}\left[\int_0^\infty \frac{e^{-\frac{zs}{\pi}}}{1+e^{-z}} dz\right] = -\frac{\pi}{2s^2} - \frac{1}{\pi}\beta'(\frac{s}{\pi})\,,
\end{aligned} \tag{C.19}$$

where we introduced the β-function by

$$\beta(x) \equiv \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right] \tag{C.20}$$

(see [52], p.331, 3.311, 2.).[22]

C.2 Leading order asymptotic of $\Im\{R_2[s, \theta, \omega]\}$

We can treat $R_2[s, \theta, \omega]$ starting from (C.13) in very much the same way as $R_1[s, \theta, \omega]$. Noting that we can again put θ equal to zero since the exponential factor $e^{2\pi y}$ guarantees the convergence of the integral, $R_2[s, 0, \omega]$ becomes explicitly:

$$\begin{aligned}
R_2[s, 0, \omega] = \int_0^\infty \frac{1}{e^{2\pi y} - 1} & \left[\frac{-y + \imath y^2}{1+y^2} e^{-ys} \frac{{}_2F_1\left[\frac{1+\imath y}{2}, \frac{1+\imath y}{2}; 1+\imath y; z(0)\right]}{{}_2F_1\left[\frac{1+\imath y}{2}, \frac{-1+\imath y}{2}; 1+\imath y; z(0)\right]}\right. \\
& \left. + \frac{-y - \imath y^2}{1+y^2} e^{ys} \frac{{}_2F_1\left[\frac{1-\imath y}{2}, \frac{1-\imath y}{2}; 1-\imath y; z(0)\right]}{{}_2F_1\left[\frac{1-\imath y}{2}, \frac{-1-\imath y}{2}; 1-\imath y; z(0)\right]}\right] dy\,.
\end{aligned} \tag{C.21}$$

[22] By β' in (C.19) we mean the derivative of β with respect to its argument.

If we now employ the expansion (C.16) two times in (C.21) we find after some algebra:

$$\Im\left\{R_2^{(0)}[s,0,\omega]\right\} = \Im\left\{\int\limits_0^\infty \frac{y}{e^{2\pi y}-1}\left[e^{-ys}\psi\left(\frac{1+\imath y}{2}\right)+e^{ys}\psi\left(\frac{1-\imath y}{2}\right)\right]dy\right\} \tag{C.22}$$

and using (C.18) we have

$$\begin{aligned}
\Im\left\{R_2^{(0)}[s,0,\omega]\right\} &= \int\limits_0^\infty \frac{\pi}{2}\frac{y}{e^{2\pi y}-1}\left[e^{-ys}\tanh\left(\frac{\pi y}{2}\right)+e^{ys}\tanh\left(-\frac{\pi y}{2}\right)\right]dy \qquad \text{(C.23)}\\
&= -\pi\int\limits_0^\infty \frac{y}{e^{2\pi y}-1}\tanh\left(\frac{\pi y}{2}\right)\sinh(ys)dy = -\pi\int\limits_0^\infty \frac{y}{(1+e^{\pi y})^2}\sinh(ys)dy\\
&= -\pi\frac{d}{ds}\left[\int\limits_0^\infty \frac{\cosh(ys)}{(1+e^{\pi y})^2}dy\right] = -\frac{1}{2}\frac{d}{ds}\left[\int\limits_0^1 \frac{u^{1-\frac{s}{\pi}}}{(1+u)^2}du+\int\limits_0^1 \frac{u^{1+\frac{s}{\pi}}}{(1+u)^2}du\right]\\
&= -\frac{1}{2}\frac{d}{ds}\left\{\frac{1}{2-\frac{s}{\pi}}\ {}_2F_1\left[2,2-\frac{s}{\pi};3-\frac{s}{\pi};-1\right]+\frac{1}{2+\frac{s}{\pi}}\ {}_2F_1\left[2,2+\frac{s}{\pi};3+\frac{s}{\pi};-1\right]\right\}.
\end{aligned}$$

While we substituted y by u according to $u=e^{-\pi y}$ in the third line, we used (3.194, p.313) of [52] with $\nu=2$, $u=\beta=1$ and $\mu=2\mp s/\pi$ (valid for $s<2\pi$) in the last line. In order to simplify the hypergeometric functions, we first use another relation between contiguous functions, namely eq. (15.2.17) of [50] with $a=1$, $b=k$, $c=k+1$ and $z=-1$ and then the formulae (15.1.21) and (15.1.23) of [50] together with the duplication formula for the Γ-function to obtain:

$$\begin{aligned}
\frac{1}{k}\ {}_2F_1[2,k;k+1;-1] &= {}_2F_1[1,k;k;-1]-\frac{k-1}{k}\ {}_2F_1[1,k;k+1;-1]\\
&= 2^{-k}\sqrt{\pi}\frac{\Gamma[k]}{\Gamma\left[\frac{k}{2}\right]\Gamma\left[\frac{k+1}{2}\right]}-(k-1)\beta(k)\\
&= \frac{1}{2}+(1-k)\beta(k). \qquad \text{(C.24)}
\end{aligned}$$

After making use of this in (C.23) we finally obtain for the contribution to lowest order in $1-z(0)$:

$$\Im\left\{R_2^{(0)}[s,0,\omega]\right\} = \frac{1}{2}\frac{d}{ds}\left[\left(1-\frac{s}{\pi}\right)\beta(2-\frac{s}{\pi})+\left(1+\frac{s}{\pi}\right)\beta(2+\frac{s}{\pi})\right]. \tag{C.25}$$

Summarizing the results to lowest order in $1-z(0)$ we therefore have

$$\begin{aligned}
\Im\left\{R_1^{(0)}[s,0,\omega]\right\} &= -\frac{\pi}{2s^2}-\frac{1}{\pi}\beta'(\frac{s}{\pi}),\\
\Im\left\{R_2^{(0)}[s,0,\omega]\right\} &= \frac{1}{2}\frac{d}{ds}\left[\left(1-\frac{s}{\pi}\right)\beta(2-\frac{s}{\pi})+\left(1+\frac{s}{\pi}\right)\beta(2+\frac{s}{\pi})\right]. \qquad \text{(C.26)}
\end{aligned}$$

Expanding (C.26) around $s=0$ we find:

$$\begin{aligned}
\Im\left\{R_1^{(0)}[s,0,\omega]\right\} &= \frac{\pi}{2s^2}-\frac{\pi}{12}+\mathcal{O}(s),\\
\Im\left\{R_2^{(0)}[s,0,\omega]\right\} &= \left(\frac{1}{6}-\frac{3\zeta(3)}{2\pi^2}\right)s+\mathcal{O}(s^3). \qquad \text{(C.27)}
\end{aligned}$$

with ζ denoting RIEMANN's ζ-function. As expected the contribution from $R_2^{(0)}[s,0,\omega]$ is subleading with respect to the one from $R_1^{(0)}[s,0,\omega]$.

C.3 Next to leading order asymptotic of $\Im\left\{R_1[s,\theta,\omega]\right\}$

We would now like to obtain the next to leading order term in the asymptotic of $\Im\left\{R_1[s,\theta,\omega]\right\}$, that is the term obtained from (C.15) by taking into account the linear contribution in $1-z(0)$ of the expansion (C.16). Since the calculation of the integral obtained in this way is rather cumbersome and anyway we do not need the full s-dependence, we do not intend to evaluate it fully. Extracting the leading s-divergence at $s=0$ is sufficient for our purposes here. To start, we insert the linear term of (C.16) in (C.15):

$$\begin{aligned}\Im\left\{R_1^{(1)}[s,0,\omega]\right\} = -\Im\Bigg\{\int_0^\infty dyy\frac{1-\imath y}{1+y^2}e^{-ys}\Bigg\{&\left(\frac{1+\imath y}{2}\right)^3\left[2\psi(2)-2\psi\left(\frac{3+\imath y}{2}\right)-\ln\left[1-z(0)\right]\right]\\ &-\left(\frac{1+\imath y}{2}\right)^2\left(\frac{-1+\imath y}{2}\right)\left[2\psi(1)-2\psi\left(\frac{1+\imath y}{2}\right)-\ln\left[1-z(0)\right]\right]\times\\ &\times\left[\ln\left[1-z(0)\right]-\psi(1)-\psi(2)+\psi\left(\frac{3+\imath y}{2}\right)+\psi\left(\frac{1+\imath y}{2}\right)\right]\Bigg\}\left[1-z(0)\right]\Bigg\}.\end{aligned} \tag{C.28}$$

After some algebra and by using the functional relation of the Digamma-function $\psi(z)$ (see [50])

$$\psi\left(z+1\right)=\psi\left(z\right)+\frac{1}{z} \tag{C.29}$$

we find

$$\Im\left\{R_1^{(1)}[s,0,\omega]\right\} = -\frac{1-z(0)}{2}\int_0^\infty ye^{-ys}\Im\left\{\ldots\right\}dy \tag{C.30}$$

with

$$\begin{aligned}\Im\left\{\ldots\right\} = \Im\Bigg\{&\left(\frac{1+\imath y}{2}\right)^2\left[2\psi(2)-2\psi\left(\frac{1+\imath y}{2}\right)-\ln\left[1-z(0)\right]\right]-(1+\imath y)\\ &+\frac{1+y^2}{4}\left[2\psi\left(1\right)-2\psi\left(\frac{1+\imath y}{2}\right)-\ln\left[1-z(0)\right]\right]\times\\ &\times\left[\ln\left[1-z(0)\right]-\psi\left(1\right)-\psi\left(2\right)+2\psi\left(\frac{1+\imath y}{2}\right)\right]\\ &+\frac{1-\imath y}{2}\left[2\psi\left(1\right)-2\psi\left(\frac{1+\imath y}{2}\right)-\ln\left[1-z(0)\right]\right]\Bigg\}.\end{aligned} \tag{C.31}$$

We will now keep only the terms proportional to y^3 in (C.30) (y^2 in (C.31)) since only these will contribute to the leading $1/s^4$ singular behavior. After a few lines of algebra we find

$$\begin{aligned}\Im\left\{R_1^{(1)}[s,0,\omega]\right\} = &-\frac{1-z(0)}{2}\left[1-2\gamma-\ln\left[1-z(0)\right]\right]\int_0^\infty y^3e^{-ys}\Im\left[\psi\left(\frac{1+\imath y}{2}\right)\right]dy\\ &+\left[1-z(0)\right]\int_0^\infty y^3e^{-ys}\Im\left[\psi\left(\frac{1+\imath y}{2}\right)\right]\Re\left[\psi\left(\frac{1+\imath y}{2}\right)\right]dy\\ &+\text{terms involving lower powers of } y.\end{aligned} \tag{C.32}$$

The first integral can be reduced to the integral (C.17) simply by replacing each power of y by a derivative with respect to s and by taking the derivatives out of the integral:

$$\begin{aligned}\int_0^\infty y^3 e^{-ys}\Im\left[\psi\left(\frac{1+\imath y}{2}\right)\right]dy &= -\frac{d^3}{ds^3}\left\{\int_0^\infty e^{-ys}\Im\left[\psi\left(\frac{1+\imath y}{2}\right)\right]dy\right\} \\ &= -\frac{\pi}{2}\frac{d^3}{ds^3}\left[-\frac{1}{s}+\frac{2}{\pi}\beta\left(\frac{s}{\pi}\right)\right]. \qquad \text{(C.33)}\end{aligned}$$

We could not find an analytic expression for the second integral. However all we need is the first term of its small s asymptotic[23] and this can be easily obtained from the large y asymptotic of $\Re\left[\psi\left((1+\imath y)/2\right)\right]$. Since asymptotically

$$\psi(z)\sim \ln z-\frac{1}{2z}-\frac{1}{12z^2}+\mathcal{O}(\frac{1}{z^4}), \qquad (z\to\infty \text{ in } |\arg z|<\pi) \qquad \text{(C.34)}$$

we expect that this term will contribute logarithmic terms in s and we find after straightforward expansions

$$\Re\left[\psi\left(\frac{1+\imath y}{2}\right)\right]\sim \ln\frac{y}{2}-\frac{1}{6y^2}+\mathcal{O}\left(\frac{1}{y^4}\right). \qquad \text{(C.35)}$$

From the last formula (C.35) we learn that within our current approximation of keeping only highest (cubic) power terms in y, it is sufficient to take into account only the contribution coming from $\ln\frac{y}{2}$. Therefore, we need to calculate the following integral

$$\frac{\pi}{2}\int_0^\infty y^3 e^{-ys}\tanh\left(\frac{\pi y}{2}\right)\ln y\, dy = -\frac{\pi}{2}\frac{d^3}{ds^3}\underbrace{\left[\int_0^\infty e^{-ys}\tanh\left(\frac{\pi y}{2}\right)\ln y\, dy\right]}_{\equiv\mathcal{J}(s)}, \qquad \text{(C.36)}$$

where we used (C.18) once more. Note that the term proportional to $\ln 2$ can be accounted for by adding a contribution of the type of the first integral in (C.32). One way to find the asymptotic of $\mathcal{J}(s)$ for small s is to integrate the following asymptotic expansion of $\tanh x$

$$\tanh x = 1-2e^{-2x}+2e^{-4x}-2e^{-6x}+\ldots = 1+2\sum_{\nu=1}^\infty(-1)^\nu e^{-2\nu x} \qquad \text{(C.37)}$$

which converges for all $x>0$. Inserting this in the definition of $\mathcal{J}(s)$ we find:

$$\begin{aligned}\mathcal{J}(s) &= \int_0^\infty e^{-ys}\ln y\left[1+2\sum_{\nu=1}^\infty(-1)^\nu e^{-\pi y\nu}\right]dy \\ &= -\frac{\gamma+\ln s}{s}+2\sum_{\nu=1}^\infty(-1)^{\nu+1}\frac{\gamma+\ln(s+\nu\pi)}{s+\nu\pi} \\ &= -\frac{\gamma+\ln s}{s}+\frac{2}{\pi}(\gamma+\ln\pi)\,\beta\left(\frac{s}{\pi}+1\right)+\frac{2}{\pi}\sum_{\nu=1}^\infty(-1)^{\nu+1}\frac{\ln\left(\nu+\frac{s}{\pi}\right)}{\nu+\frac{s}{\pi}}, \qquad \text{(C.38)}\end{aligned}$$

[23] In the sense $s\ll 1$.

where we used the series expansion of the β-function given e.g. in [52]. The last two terms in the above formula are finite in the limit $s \to 0$ and therefore will play no role in the asymptotic (since they are multiplied by a factor of $1 - z(0)$).

We are now ready to assemble all contributions to the small s asymptotic of $\Im\left\{R_1^{(1)}[s,0,\omega]\right\}$:

$$\Im\left\{R_1^{(1)}[s,0,\omega]\right\} \sim \frac{1-z(0)}{2}\left[1-2\gamma-\ln\left[1-z(0)\right]+2\ln 2\right]\frac{\pi}{2}\frac{d^3}{ds^3}\left[-\frac{1}{s}+\frac{2}{\pi}\beta\left(\frac{s}{\pi}\right)\right]$$
$$-\left[1-z(0)\right]\frac{\pi}{2}\frac{d^3}{ds^3}\left[-\frac{\gamma+\ln s}{s}+\frac{2}{\pi}\left(\gamma+\ln\pi\right)\beta\left(\frac{s}{\pi}+1\right)+\frac{2}{\pi}\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}\ln\left(\nu+\frac{s}{\pi}\right)}{\nu+\frac{s}{\pi}}\right]+\mathcal{O}\left(\frac{\ln s}{s^3}\right)$$
$$=\frac{\pi}{2}\frac{1-z(0)}{s^4}\left\{8-6\ln 2-6\ln\left[\frac{s}{\sqrt{1-z(0)}}\right]\right\}+\mathcal{O}\left(\frac{\ln s}{s^3}\right), \qquad \text{(C.39)}$$

where we used the following series expansions for the β-function and for the last term in (C.38):

$$\beta\left(\frac{s}{\pi}\right) = \frac{\pi}{s} - \ln 2 + \mathcal{O}(s)\,,$$
$$\beta\left(\frac{s}{\pi}+1\right) = \ln 2 + \mathcal{O}(s)\,,$$
$$\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}\ln\left(\nu+\frac{s}{\pi}\right)}{\nu+\frac{s}{\pi}} = \frac{1}{2}\left[\ln(2)\right]^2 - \gamma\ln(2) + \mathcal{O}(s). \qquad \text{(C.40)}$$

Note that only the term $\beta(s/\pi)$ gave a contribution to the leading $1/s^4$ divergence. Relations (C.26) and (C.39) constitute the main result of this appendix. They provide the desired asymptotic behavior of the two-point function in the regime of intermediate distances $\omega/\sinh\left(\frac{\omega\pi}{2}\right) \ll s \ll 1$. A couple of remarks are in place. Clearly, (C.26) is not only an asymptotic result but is valid for all $s \in [0,\pi]$. Secondly, we were writing $\mathcal{O}(\ln s/s^3)$ in (C.39) to indicate that we dropped all contributions coming from terms in the integrand (C.31) lower than cubic order. Finally we point out that the development of the ratio (C.16) in power of $1-z(0)$ indeed generates an asymptotic in the correct distance regime. This is clear from the fact that each new power in $1-z(0)$ is paired with an additional power of y^2 in the integrand of $\Im\left\{R_1[s,\theta,\omega]\right\}$. This by itself implies upon multiplication by e^{-ys} and integration over y an additional power of $1/s^2$. We therefore effectively generate an expansion in powers of $\sqrt{1-z(0)}/s = 1/\left[s\cosh\left(\frac{\omega\pi}{2}\right)\right]$. It is also easy to understand why this rough way of counting powers works. The reason is that the asymptotic properties of our integrals $\Im\left\{R_1^{(n)}[s,0,\omega]\right\}$ are determined only by the behavior of the corresponding integrands for large values of the integration variable y. The terms $\Re\left[\psi\left((1+\imath y)/2\right)\right]$ and $\Im\left[\psi\left((1+\imath y)/2\right)\right]$ do not interfere with the above power counting since their behavior for large y is either logarithmic in y (for $\Re$) or constant in y (for $\Im$).

Finally we want to point out that the expansion in powers of $1/\left[s\cosh\left(\frac{\omega\pi}{2}\right)\right]$ is not useful for obtaining information about the Green's function for distances $s \ll \omega/\cosh\left(\frac{\omega\pi}{2}\right)$. In this limit the asymptotic expansion of the hypergeometric functions in (C.2) for large values of λ proves the most efficient way to recover the properties of a Green's function in a 4-dimensional space. See appendix D for the detailed form of the Green's function at distances smaller than the extra dimension.

C.4 Estimate of the arc contribution to $R_1[s,\theta,\omega]$

This final subsection is dedicated to the verification that the arc denoted C_2 in Fig. 3 of appendix A gives a vanishing contribution to the integral $R_1[s,\theta,\omega]$ and thus justifying the representation (C.15). We need to know the large λ asymptotic behavior of the hypergeometric functions entering the definition of $R_1[s,\theta,\omega]$ in (C.12). The relevant formula (D.1) can be found in appendix D where we discuss the short distance behavior of the Green's function. After canceling all common factors from the ratio of the two hypergeometric functions we find:

$$\frac{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda+1}{2};\lambda+1;z(\theta)\right]}{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda-1}{2};\lambda+1;z(0)\right]}=\frac{\lambda+1}{\lambda-1}\left[\frac{z(\theta)}{z(0)}\right]^{-\frac{\lambda+1}{2}}\left[\frac{e^{-\nu(\theta)}}{e^{-\nu(0)}}\right]^{\frac{\lambda+1}{2}}\frac{\left[1+e^{-\nu(\theta)}\right]^{-\frac{1}{2}}\left[1-e^{-\nu(\theta)}\right]^{-\frac{1}{2}}}{\left[1+e^{-\nu(0)}\right]^{\frac{3}{2}}\left[1-e^{-\nu(0)}\right]^{\frac{1}{2}}}\times$$
$$\times\left[1+\mathcal{O}\left(\frac{1}{\lambda}\right)\right]. \tag{C.41}$$

Substituting $\lambda=Re^{\imath\varphi}$, we now estimate $R_1[s,\theta,\omega]$:

$$\left|\int_{C_2}\frac{\lambda e^{\imath\lambda s}}{\lambda+1}\left[\frac{z(\theta)}{z(0)}\right]^{\frac{\lambda}{2}}\frac{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda+1}{2};\lambda+1;z(\theta)\right]}{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda-1}{2};\lambda+1;z(0)\right]}d\lambda\right|$$
$$\le\int_{C_2}\left|\frac{\lambda e^{\imath\lambda s}}{\lambda+1}\left[\frac{z(\theta)}{z(0)}\right]^{\frac{\lambda}{2}}\frac{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda+1}{2};\lambda+1;z(\theta)\right]}{{}_2F_1\left[\frac{\lambda+1}{2},\frac{\lambda-1}{2};\lambda+1;z(0)\right]}\right|d\lambda$$
$$\le\underbrace{C\left[\frac{z(0)}{z(\theta)}\right]^{\frac{1}{2}}\left|\frac{e^{-\nu(\theta)}}{e^{-\nu(0)}}\right|^{\frac{1}{2}}\frac{\left[1+e^{-\nu(\theta)}\right]^{-\frac{1}{2}}\left[1-e^{-\nu(\theta)}\right]^{-\frac{1}{2}}}{\left[1+e^{-\nu(0)}\right]^{\frac{3}{2}}\left[1-e^{-\nu(0)}\right]^{\frac{1}{2}}}}_{K}R\int_0^{\frac{\pi}{2}}\left|e^{\imath Rse^{\imath\varphi}}\right|\left|\frac{Re^{\imath\varphi}}{Re^{\imath\varphi}-1}\right|\left|\left[\frac{e^{-\nu(\theta)}}{e^{-\nu(0)}}\right]^{\frac{Re^{\imath\varphi}}{2}}\right|d\varphi$$
$$=KR\int_0^{\frac{\pi}{2}}\frac{e^{-R(s\sin\varphi-\ln a\cos\varphi)}}{\sqrt{1-\frac{2\cos\varphi}{R}+\frac{1}{R^2}}}d\varphi\le K'R\int_0^{\frac{\pi}{2}}e^{-Rs\left(\sin\varphi+\frac{\ln a^{-1}}{s}\cos\varphi\right)}d\varphi\,, \tag{C.42}$$

where C is a constant independent of R and the estimate of the ratio of the two hypergeometric functions is valid for large enough R. We also introduced a as in appendix D by (D.7). Note that $a<1$ for $0<\theta$. In the last line we estimated the inverse of the square root by an arbitrary constant (e.g. 2), certainly good for large enough R and for all φ. All what is left is to observe that

$$0<\epsilon\equiv\min[1,\frac{\ln a^{-1}}{s}]\le\sin\varphi+\frac{\ln a^{-1}}{s}\cos\varphi\quad\text{for}\quad s>0,\;\;a^{-1}>1 \tag{C.43}$$

to obtain the desired result:

$$\left|\int_{C_2}d\lambda\ldots\right|\le K'R\int_0^{\frac{\pi}{2}}e^{-Rs\epsilon}d\varphi=\frac{\pi}{2}K'Re^{-Rs\epsilon}\to 0\quad\text{for}\quad R\to\infty. \tag{C.44}$$

D The two point function at very short distances ($r \ll R$)

The aim of this appendix is to calculate the behavior of the two-point function at distances smaller than the extra dimension $r \ll R$. We will do so by using an appropriate asymptotic expansion for large parameter values of the hypergeometric functions in (C.2). Such an asymptotic can be found for example in [53] or [54]. The general formula is[24]

$$
\begin{aligned}
{}_2F_1\left[a+n, a-c+1+n; a-b+1+2n; z\right] = \frac{2^{a+b}\Gamma[a-b+1+2n]\left(\frac{\pi}{n}\right)^{\frac{1}{2}}}{\Gamma[a-c+1+n]\Gamma[c-b+n]} \frac{e^{-\nu(n+a)}}{z^{(n+a)}} \times \\
\times \left(1+e^{-\nu}\right)^{\frac{1}{2}-c}\left(1-e^{-\nu}\right)^{c-a-b-\frac{1}{2}}\left[1+\mathcal{O}\left(\frac{1}{n}\right)\right]
\end{aligned} \tag{D.1}
$$

with ν defined via the relation

$$
e^{-\nu} = \frac{2-z-2(1-z)^{\frac{1}{2}}}{z}. \tag{D.2}
$$

We need to calculate

$$
S[s,\theta,\omega] = \sum_{\lambda=2}^{\infty} \lambda \sin(\lambda s) \frac{\varphi_1^{(\lambda)}(\theta)}{\varphi'^{(\lambda)}_1(0)} \tag{D.3}
$$

with the ratio $\varphi_1^{(\lambda)}(\theta)/\varphi'^{(\lambda)}_1(0)$ given in (C.2). Using the above expansion (D.1) two times we find after numerous cancellations

$$
\begin{aligned}
S[s,\theta,\omega] \sim -\frac{4}{\omega}\left(\frac{1-z(\theta)}{1-z(0)}\right)^2 \frac{\left[1+e^{-\nu(\theta)}\right]^{-\frac{1}{2}}\left[1-e^{-\nu(\theta)}\right]^{-\frac{5}{2}}}{\left[1+e^{-\nu(0)}\right]^{\frac{1}{2}}\left[1-e^{-\nu(0)}\right]^{-\frac{3}{2}}} \frac{z(0)^{\frac{3}{2}}}{z(\theta)^2} \frac{e^{-\frac{3\nu(\theta)}{2}}}{e^{-\frac{\nu(0)}{2}}} \times \\
\times \sum_{\lambda=2}^{\infty} \frac{\lambda \sin(\lambda s)}{\lambda-1}\left[\frac{e^{-\nu(\theta)}}{e^{-\nu(0)}}\right]^{\frac{\lambda}{2}},
\end{aligned} \tag{D.4}
$$

where we used the following abbreviations

$$
z(\theta) = \tanh^2\left[\omega\left(\frac{\pi}{2}-\theta\right)\right], \tag{D.5}
$$

$$
e^{-\nu(\theta)} = \tanh^2\left[\frac{\omega}{2}\left(\frac{\pi}{2}-\theta\right)\right]. \tag{D.6}
$$

Note that (D.6) is obtained after some lines of algebra by inserting (D.5) in (D.2). Still the sum in (D.4) diverges for $\theta = 0$ but it turns out that it can be performed analytically for $\theta > 0$ so that after the summation the limit $\theta \to 0$ exists. For the sake of a lighter notation we introduce the symbol a by

$$
a \equiv \left[\frac{e^{-\nu(\theta)}}{e^{-\nu(0)}}\right]^{\frac{1}{2}} = \frac{\tanh\left[\frac{\omega}{2}\left(\frac{\pi}{2}-\theta\right)\right]}{\tanh\left(\frac{\omega\pi}{4}\right)}. \tag{D.7}
$$

[24] Since we noticed that the formulae given by Luke, p. 452 eq. (20)–(23) [54] and the formula given by Erdélyi [53] differ by a factor of 2^{a+b} we performed a numerical check. Its outcome clearly favored Erdélyi's formula which we reproduced in D.1.

It is now straightforward to evaluate the sum over λ:

$$\begin{aligned}
\sum_{\lambda=2}^{\infty} \frac{\lambda \sin(\lambda s)}{\lambda - 1} a^{\lambda} &= \sum_{\lambda=2}^{\infty} \sin(\lambda s)\, a^{\lambda} + \sum_{\lambda=2}^{\infty} \frac{\sin(\lambda s)}{\lambda - 1} a^{\lambda} \\
&= -a \sin s + \sum_{\lambda=1}^{\infty} \sin(\lambda s)\, a^{\lambda} + \sum_{\lambda=1}^{\infty} \frac{\sin[(\lambda+1)\, s]}{\lambda} a^{\lambda+1} \\
&= -a \sin s + \frac{1}{2} \frac{\sin s}{\frac{1}{2}\left(a + \frac{1}{a}\right) - \cos s} - \frac{a}{2} \sin s \, \ln\left(1 - 2a\cos s + a^2\right) \\
&\quad + a \cos s \arctan\left[\frac{a \sin s}{1 - a\cos s}\right] .
\end{aligned} \tag{D.8}$$

The full result can therefore be written as

$$\begin{aligned}
S[s,\theta,\omega] \sim -\frac{4}{\omega}\left(\frac{1 - z(\theta)}{1 - z(0)}\right)^2 \frac{\left[1 + e^{-\nu(\theta)}\right]^{-\frac{1}{2}} \left[1 - e^{-\nu(\theta)}\right]^{-\frac{5}{2}}}{\left[1 + e^{-\nu(0}\right]^{\frac{1}{2}} \left[1 - e^{-\nu(0)}\right]^{-\frac{3}{2}}} \frac{z(0)^{\frac{3}{2}}}{z(\theta)^2} \frac{e^{-\frac{3\nu(\theta)}{2}}}{e^{-\frac{\nu(0)}{2}}}\, a \sin s \times \\
\times \left\{ \frac{1}{2a} \frac{1}{\frac{1}{2}\left(a + \frac{1}{a}\right) - \cos s} - \frac{1}{2} \ln\left(1 - 2a\cos s + a^2\right) + \cot s \, \arctan\left[\frac{a \sin s}{1 - a\cos s}\right] - 1 \right\} .
\end{aligned} \tag{D.9}$$

Although the last result is given in closed form, we have to emphasize that its validity is restricted to distances smaller than the size of the extra dimension R.

It is now save to take the limit $\theta \to 0$ in (D.9). The result is

$$\lim_{\theta \to 0} S[s,\theta,\omega] = -\frac{\sinh\left(\frac{\omega\pi}{2}\right)}{\omega} \sin s \left\{ \frac{1}{4} \frac{1}{\sin^2\left(\frac{s}{2}\right)} - \ln\left[2 \sin\left(\frac{s}{2}\right)\right] + \frac{\pi - s}{2} \cot s \; - 1 \right\} . \tag{D.10}$$

We note that it is the first term in the curly brackets on the right hand side of (D.9) which is responsible for reproducing the characteristic short distance singularity of the two-point function. The main result of this appendix is therefore:

$$\lim_{\theta \to 0} S[s,\theta,\omega] \sim -\frac{\sinh\left(\frac{\omega\pi}{2}\right)}{\omega} \left\{ \frac{1}{s} + \frac{\pi}{2} + \mathcal{O}\left(s \ln s\right) \right\} . \tag{D.11}$$

The 5-dimensional short distance singularity can even be obtained including the extra dimension. Apart from the logarithmic term in the curly brackets of (D.9), all but the first one are finite at short distances (in θ and s). Developing the denominator of this term, we recover the usual euclidean metric in $\mathbb{R}^4$:

$$\frac{1}{2}\left(a + \frac{1}{a}\right) - \cos s \sim \frac{1}{2}\left[\frac{\omega^2}{\sinh^2\left(\frac{\omega\pi}{2}\right)} \theta^2 + s^2\right] + \ldots . \tag{D.12}$$

where the dots denote higher terms in θ and s.

References

[1] I. Antoniadis, Phys. Lett. B **246**, 377 (1990).

[2] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **429**, 263 (1998) [arXiv:hep-ph/9803315].

[3] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **436**, 257 (1998) [arXiv:hep-ph/9804398].

[4] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B **125**, 136 (1983).

[5] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) [arXiv:hep-ph/9905221].

[6] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [arXiv:hep-th/9906064].

[7] A. G. Cohen and D. B. Kaplan, Phys. Lett. B **470**, 52 (1999) [arXiv:hep-th/9910132].

[8] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B **125**, 139 (1983).

[9] C. Wetterich, Nucl. Phys. B **255**, 480 (1985).

[10] S. Randjbar-Daemi and C. Wetterich, Phys. Lett. B **166**, 65 (1986).

[11] J. Polchinski, Phys. Rev. Lett. **75**, 4724 (1995) [arXiv:hep-th/9510017].

[12] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921**, 966 (1921).

[13] O. Klein, Z. Phys. **37**, 895 (1926) [Surveys High Energ. Phys. **5**, 241 (1986)].

[14] A. Salam and J. Strathdee, Annals Phys. **141**, 316 (1982).

[15] *Modern Kaluza-Klein Theories*, eds. T. Appelquist, A. Chodos and P. G. Freund, (Addison-Wesley, 1987).

[16] V. A. Rubakov, Phys. Usp. **44**, 871 (2001) [Usp. Fiz. Nauk **171**, 913 (2001)] [arXiv:hep-ph/0104152].

[17] W. Muck, K. S. Viswanathan and I. V. Volovich, Phys. Rev. D **62**, 105019 (2000) [arXiv:hep-th/0002132].

[18] R. Gregory, V. A. Rubakov and S. M. Sibiryakov, Class. Quant. Grav. **17**, 4437 (2000) [arXiv:hep-th/0003109].

[19] A. Chodos and E. Poppitz, Phys. Lett. B **471**, 119 (1999) [arXiv:hep-th/9909199].

[20] I. Olasagasti and A. Vilenkin, Phys. Rev. D **62**, 044014 (2000) [arXiv:hep-th/0003300].

[21] I. Olasagasti, Phys. Rev. D **63**, 124016 (2001) [arXiv:hep-th/0101203].

[22] R. Gregory, Phys. Rev. Lett. **84**, 2564 (2000) [arXiv:hep-th/9911015].

[23] T. Gherghetta and M. E. Shaposhnikov, Phys. Rev. Lett. **85**, 240 (2000) [arXiv:hep-th/0004014].

[24] T. Gherghetta, E. Roessl and M. E. Shaposhnikov, Phys. Lett. B **491**, 353 (2000) [arXiv:hep-th/0006251].

[25] S. Y. Khlebnikov and M. E. Shaposhnikov, Phys. Lett. B **203**, 121 (1988).

[26] C. W. Misner, K. S. Thorne and J. A. Wheeler, "Gravitation", Freeman and Company, San Francisco (1973)

[27] M. Visser, Phys. Lett. B **159**, 22 (1985) [arXiv:hep-th/9910093].

[28] C. Csaki, J. Erlich and C. Grojean, Nucl. Phys. B **604**, 312 (2001) [arXiv:hep-th/0012143].

[29] S. L. Dubovsky, JHEP **0201**, 012 (2002) [arXiv:hep-th/0103205].

[30] S. R. Coleman and S. L. Glashow, Phys. Rev. D **59**, 116008 (1999) [arXiv:hep-ph/9812418].

[31] C. D. Hoyle, U. Schmidt, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, D. J. Kapner and H. E. Swanson, Phys. Rev. Lett. **86**, 1418 (2001) [arXiv:hep-ph/0011014].

[32] R. M. Wald, *General Relativity*, The university of Chicago Press, Chicago and London, 1984.

[33] D. Youm, Mod. Phys. Lett. A **16**, 2371 (2001) [arXiv:hep-th/0110013].

[34] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[35] S.W.Hawking, G. Ellis, *The large scale structure of Space-time*; Cambridge Univ. Press, London, 1973

[36] S. J. Avis, C. J. Isham and D. Storey, Phys. Rev. D **18**, 3565 (1978).

[37] M. A. Rubin and C. R. Ordonez, J. Math. Phys. **26**, 65 (1985).

[38] B. Allen and M. Turyn, Nucl. Phys. B **292**, 813 (1987).

[39] S. Mukohyama, Phys. Rev. D **62**, 084015 (2000) [arXiv:hep-th/0004067].

[40] W. Israel, Nuovo Cim. B **44S10**, 1 (1966) [Erratum-ibid. B **48**, 463 (1967 NUCIA,B44,1.1966)].

[41] I. Stakgold, *Green's functions and boundary value problems*; John Wiley & Sons, 1998

[42] H. Kodama, A. Ishibashi and O. Seto, Phys. Rev. D **62**, 064022 (2000) [arXiv:hep-th/0004160].

[43] A. A. Grib, S. G. Mamayev, V. M. Mostepanenko *Vacuum quantum effects in strong fields*, Friedmann Laboratory publishing, St.Petersburg 1994

[44] D. Astefanesei and E. Radu, Grav. Cosmol. **7**, 165 (2001).

[45] B. C. Nolan, Class. Quant. Grav. **16**, 3183 (1999) [arXiv:gr-qc/9907018].

[46] P. Callin and F. Ravndal, arXiv:hep-ph/0403302.

[47] E. Kiritsis, N. Tetradis and T. N. Tomaras, JHEP **0203**, 019 (2002) [arXiv:hep-th/0202037].

[48] K. Ghoroku, A. Nakamura and M. Yahiro, Phys. Lett. B **571**, 223 (2003) [arXiv:hep-th/0303068].

[49] S. B. Giddings, E. Katz and L. Randall, JHEP **0003**, 023 (2000) [arXiv:hep-th/0002091].

[50] M. Abramowitz, I. A. Stegun *Handbook of mathematical functions*, Dover Publications, Inc., New York, 1972

[51] F. W. J. Olver *Asymptotics and special functions*, Academic Press, New York, London, 1974

[52] I. S. Gradshteyn, I. M. Ryzhik *Tables of integrals, series, and products*, Sixth Edition, Academic Press, 2000

[53] A. Erdélyi,W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions*, Volume I, Bateman Manuscript Project, McGraw-Hill, 1953

[54] Y. L. Luke, *Mathematical Functions and their Approximations*, Academic Press, New York, San Francisco, London, 1975